Properly taught, mathematics produces a sense of security in one branch at least of human thought that is not subject to the vicissitudes of time. All the philosophy connected with mathematics should be mentioned at the proper time and made to serve as a vitalizing element in the course. The beauty in mathematics should be dwelt upon, and exhibited frequently. This kind of a course minimizes the tactical or manipulative work as far as possible in order to get the student acquainted with the deep principles of the subject. The girl's imaginative, inventive, and intuitive ability should be cultivated to the utmost, as being the best thing for her development.

4. The paper by Professor Garabedian was in two parts. In the first section, devoted to a brief survey of the literature, it was pointed out that contributions to the subject "Music and Mathematics" have been made chiefly by the professional musician or the professional mathematician, and frequently give evidence of the bias of the specialist. Next to nothing has been written from that comprehensive point of view in which the writer, disparaging neither art in favor of the other, inquires, with love for both, if there be not perhaps certain bonds of kinship between the two. In the second section of the paper, music and mathematics were considered as two arts eminently deserving of comparative study; attention was called to certain similarities and dissimilarities, and some suggestive questions were raised—and answers attempted—with reference to the human significance and philosophical meaning of music and mathematics.

5. Most of our Analytic Geometry textbooks give the equation of the ellipse $(x^2/a^2) + (y^2/a^2) = 1$ in the parametric form $x = a \cos \theta$, $y = b \sin \theta$, and state that the coördinates of any point of the ellipse may be obtained by the intersection of the lines drawn parallel to the x- and y-axis through the points where the same radius cuts the circles of radius a and b respectively. The finding of the line values of different trigonometric expressions may be applied to the construction of the remaining conics as well as many of the standard curves in both rectangular and polar coördinates.

6. Professor Pettit considered a generalization of the ordinary projective construction of a conic, obtained by relating two pencils of lines by means of two perspective curves of order m and n respectively. The intersections of corresponding rays are points of a curve of order $2mn$ having two mn-fold points, m n-fold points, n m-fold points and $mn(mn - m - n + 1)$ nodes as well as a node corresponding to each node of the base curves. If one base curve is a straight line and the other a rational curve of order n, the resulting curve is rational, of order $2n$ and has 3 n-fold points. By proper choice of the base curves and pencils certain reductions can be introduced, among which are, notably, the ordinary construction of the ellipse, the rational cubic, and the rational sextic with ten distinct nodes. The application of the principle of duality leads directly to the construction of corresponding class curves.

7. Let A, B, C, D be four points in the complex plane associated with the complex numbers a, b, c, d. Put $(abcd) = \lambda = re^{i\varphi}$. The angle φ is the angle which the tangent to the circle ABD makes with the tangent to the circle ABC

at B. With A, B, C and λ given, D may be constructed with ruler and compasses. It will be one of the two intersections of two circles; the second intersection, D', satisfies $(ABCD') = re^{i(\varphi-\psi)}$.

Assuming two points, A and B, as fixed and calling two variable points Z and W, Dr. Logsdon discussed some interesting properties of the conformal representation established by $(abzw) = \lambda$ for a fixed complex λ.

8. The paper of Professor Sellew originated in a desire to answer some questions raised in the Chicago meeting of the Society a year ago, with regard to the number of curves used by Professor Karl Pearson in fitting statistical data.

Earlier work suggested the differential equation

$$\frac{1}{y}\frac{dy}{dx} = \frac{\psi(x)}{\varphi(x)},$$

where the functions $\psi(x)$ and $\varphi(x)$ are to be determined. For unimodal curves,

$$\psi(x) = x + a.$$

Professor Pearson gives his reasons for using only four moments and the simple development

$$\varphi(x) = c_0 + c_1x + c_2x^2.$$

The character of the roots of this quadratic determines twelve types, but of these three types are of unusual importance. The other types are limiting or transitional types as the character of the statistics changes from one of these main types to the others.

The particular type of curve to be used in any case is easily obtained by calculating the fundamental constants β_1 and β_2, functions of the moments.

9. An important problem in statistical theory is that of determining a straight line giving "the best fit" to a set of points not collinear. Professor Moulton discussed the solution when "the best fit" is determined by making the sum of the squares of the perpendicular distances from the points to a line a minimum. He showed how to apply the well-known theory of the ellipsoid of inertia to this problem and discussed advantages and disadvantages of this definition of "the best fit" as compared with the one more commonly used.

10. Starting with the notion of measuring individuality by the position of an individual on the Gaussian scale corresponding to the frequency distribution of the population to which the individual belongs, Professor Crathorne considered what might be called a weighted correlation problem. A specific example is that of grading the contestants in a stock-judging contest. Each contestant ranks the animals to be judged. Placing the best animal in first place should count more than placing the second best animal in second place. Placing the second correctly should count more than placing the third best correctly, and so on. A theory was outlined for grading contestants based on a rank correlation in which the ordinal numbers giving the ranks were replaced by the numbers on

the Gaussian scale corresponding to the individual animals considered as the extreme individuals in a frequency distribution of a large population.

BESSIE I. MILLER, *Secretary-Treasurer.*

THE MAY MEETING OF THE INDIANA SECTION.

The second meeting of the Indiana Section of the Mathematical Association of America was held May 8–9, 1925, at Indiana University, Bloomington, in connection with the visit of Professor J. HADAMARD of Paris to this institution.

There were forty-three present including the following thirty members of the Association: R. J. Aley, Gladys L. Banes, C. F. Barr, J. C. Bennett, E. M. Berry, H. T. Davis, S. C. Davisson, C. S. Doan, J. E. Dotterer, W. E. Edington, E. D. Grant, G. H. Graves, J. Hadamard, L. Hadley, U. S. Hanna, C. T. Hazard, Cora B. Hennel, F. H. Hodge, E. N. Johnson, J. J. Knox, Florence Long, Juna M. Lutz, Wm. Marshall, T. E. Mason, G. E. Moore, C. K. Robbins, D. A. Rothrock, K. P. Williams, H. E. Wolfe, H. N. Wright.

On Friday evening the visiting members were present at a joint dinner of the Indiana and Purdue sections of the American Association of University Professors.

At eight o'clock Professor Hadamard gave an address under the auspices of the Indiana chapter of Sigma Xi on the subject: "The equilibrium of the solar system, past and future." The introduction was made by Professor S. C. Davisson of Indiana University.

Professor Hadamard first discussed the nature of the motions of the planets of the solar system and pointed out that the greatest cause for perturbations in the orbits of the planets is the mass of Jupiter. Although these disturbances are largely compensating, there still remains a small residual disturbance which may ultimately lead to the disintegration of the solar system. The speaker showed how the study of this problem can be based upon geometrical analogy.

The program Saturday morning consisted of an address by Professor Hadamard on the subject: "The modern notion of a function." The speaker first showed how the notion of functional relationship soon outgrew Euler's definition of "functio continua." He then pointed out that analytic functions, as a class, possess three fundamental properties. First: they can be represented by means of formal expressions. Second: they possess the group property, *i.e.*, that an analytic function of an analytic function is again an analytic function. Third: they possess the property of continuation. The speaker then inquired whether other classes of functions might not exist which would fail to have one or more of these properties, but would still possess the others. Such a class is furnished by those functions which admit the limitation $|f^{(n)}| < k^n(an)!\quad a > 1$. Functions thus characterized arise in the parabolic case of the boundary-value problem of partial differential equations and are called *quasi-analytic.* As a class, they fail to possess the third property of analytic functions. The work of Gevrey, Denjoy and Carleman was quoted.

After a short business session presided over by the chairman, Professor F. H. Hodge of Purdue University, adjournment was taken to the university cafeteria where a luncheon was served to the members and their guests.

The afternoon program consisted of the following papers:

1. "Characteristic errors in the algebra of college freshmen" by Mr. C. F. BARR of Purdue University.

2. "A simple demonstration of some lemmas of Jensen" by Professor K. P. WILLIAMS of Indiana University.

3. "What college teachers can do to improve the quality of mathematics in the secondary schools" by Professor J. E. DOTTERER of Manchester College.

4. "The solution of triangles from an algebraic point of view" by Professor C. K. ROBBINS of Purdue University.

5. "Projective geometry as an undergraduate study" by Professor H. N. WRIGHT of Earlham College.

6. "The history and present status of integral equations" by Professor H. T. DAVIS of Indiana University.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Mr. Barr presented the results of a statistical study of characteristic errors in the algebra of college freshmen based on the results of an examination of several hundred semester examination papers. The solution of linear simultaneous equations in two unknowns appears to be the easiest problem for freshmen, while the simplification of continued fractions and the determination of parameters present the greatest difficulty.

2. In his paper Professor Williams gave a simple and direct demonstration of the lemmas that form the basis for some of Jensen's theorems on analytic functions.

3. Professor Dotterer made a plea for a better supervision of the preparation of high school teachers of mathematics and suggested the coöperation of the Indiana Section of the Association with the State Board of Education. Upon motion, the chairman was instructed to appoint a committee to study the problem and present suggestions to the state board.

4. In the case of plane triangles Professor Robbins assumed the law of sines and then used the system: (1) $b/a = \sin B/\sin A$, (2) $c/a = \sin C/\sin A$, (3) $A + B + C = \pi$, to obtain by purely algebraic means the other formulas used in plane trigonometry. A similar derivation of the formulas of spherical triangles was made.

5. Professor Wright stated that an examination of the courses of study of a number of the leading colleges and universities of the country shows that courses for undergraduates in synthetic projective geometry are offered in about half of them. In this subject, better than any place in the usual undergraduate course, there is available to the student of elementary mathematics a most interesting presentation of some of the great concepts and unifying principles of geometry.

It should be included in the training of all prospective teachers and majors in mathematics.

6. The paper of Professor Davis, based upon a bibliography of 500 titles, traced the development of integral equations during the twenty-five-year period since the appearance of Fredholm's first paper in 1900.

The time and place of the next meeting were left for the decision of the executive committee.

H. T. DAVIS, *Secretary.*

THE DECEMBER MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION.

The sixteenth regular meeting of the Maryland-Virginia-District of Columbia Section was held at Johns Hopkins University, Baltimore, Md., on December 6, 1924. The members were the guests of the University at luncheon. There were two sessions; Dr. F. D. Murnaghan presided at each session.

The attendance was forty-five including the following thirty-two members of the Association: O. S. Adams, R. N. Ashmun, H. G. Avers, Clara L. Bacon, W. W. Bigelow, G. A. Bingley, C. C. Bramble, J. A. Bullard, P. Capron, G. R. Clements, A. Cohen, J. B. Eppes, H. Gwinner, W. M. Hamilton, L. S. Hulburt, W. D. Lambert, A. E. Landry, Florence P. Lewis, E. S. Mayer, F. D. Murnaghan, J. R. Musselman, C. A. Nelson, C. H. Rawlins, Jr., J. N. Rice, H. M. Robert, Jr., A. C. Robinson, H. A. Robinson, R. E. Root, G. A. Ross, J. B. Scarborough, Elizabeth W. Wilson, E. W. Woolard.

The following nine papers were read:

(1) "The slighted pyramid" by Professor HARRY GWINNER, Vice Dean, University of Maryland.

(2) "Needs of symbols for partial and total differential coefficients" by Professor A. COHEN, Johns Hopkins University.

(3) "On the Jonquières curve" by Dr. C. A. NELSON, Johns Hopkins University.

(4) "The distance between two points on an ellipsoidal earth" by Mr. W. D. LAMBERT, U. S. Coast and Geodetic Survey.

(5) "The summation method for the determination of the Pearsonian coefficient of correlation" by Miss ELIZABETH W. WILSON of the Central High School, Washington, D. C.

(6) "Parametric equations of the perimeter of regular polygons" by Dr. O. S. ADAMS, U. S. Coast and Geodetic Survey.

(7) "A geometrical discussion of right-angled triangles with integral sides" by Mr. R. L. CARY of Baltimore, Md.

(8) "A laboratory course in mathematics" by Professor R. E. ROOT, U. S. Naval Academy.

(9) "A mathematical instruments laboratory" by Professor C. C. BRAMBLE, U. S. Naval Academy.

The papers led to considerable discussion. Abstracts follow below, the numbers corresponding to the numbers in the list of titles.

1. Professor Gwinner called attention to pyramids other than those known in elementary solid geometry as regular pyramids, reference being made especially to pyramids whose bases are not regular polygons and whose vertices do not lie in a line erected to the center of the base. For want of a better name, these were designated as oblique irregular pyramids.

The properties of such pyramids are studied very elegantly by the methods of spherical trigonometry. Attention was called also to articles in the *Engineering News* by C. A. P. Turner and Hart Vance as well as in the text of Carlton T. Bishop on *Structural Details of Hip and Valley Rafters*.

The solution of the oblique irregular pyramid is made by spherical trigonometry but the details of the primary plane are somewhat more tedious than in the case of the regular pyramid. Relative to the regular pyramid, it is an instructive exercise to present to a class in spherical trigonometry the problem of finding the dihedral angles between adjacent lateral faces having given the sides of the base and the altitude. Let the student decide as to whether he shall place the vertex of the pyramid at the center of the sphere; or place one corner of the base at the center. If the proper conception is made, the usual spherical triangle is made in either case and the solution follows. Wentworth's *Trigonometry*, Crockett's *Trigonometry*, and Fletcher Durell's *Trigonometry* have given this matter some attention. This problem of the dihedral angle of the pyramid is handled very nicely in the solid portion of analytic geometry.

2. In this communication, Dr. Cohen suggests a convenient notation for derivatives of a function of several variables, intermediate between its partial derivative with respect to one of its variables when all the remaining variables are considered constant, and its total derivative with respect to that variable, when all the other variables are considered functions of it.

3. Dr. Nelson discussed a suggestion of Fubini's for an extension of the concept of the flex-line of a rational plane cubic to any rational plane curve of order n with a multiple point of order $n - 1$ (Jonquières curve). He found the conditions under which this line may be found uniquely and the limiting position assumed by the line in the alternate case.

4. Mr. Lambert's paper dealt with the problem of finding the distance between two points on an ellipsoidal earth, the two points being given by their latitudes and longitudes. Many solutions of this problem have been given to meet various conditions of distance, accuracy or convenience in computation. When the flattening of the earth is allowed for, there are various connecting lines which might be used, but the most natural one is the shortest or geodesic line. The complete solution for the geodesic was given by Bessel in 1825. It involves elliptic integrals and is decidedly laborious. Associated with the spheroidal triangle, the vertices of which are a pole of the earth and the two points, there is a spherical triangle somewhat resembling the spheroidal one and related to it in certain rather simple ways. The spherical triangle should be

conceived as a convenient geometrical representation of certain quantities occurring in the computation rather than as a deformation or projection of the spheroidal triangle. The rigorous solution involves successive approximations each requiring the solution of spherical triangles. If, however, too great accuracy is not demanded, the calculation may be made by applying to the first approximation certain small corrections deduced from the differential variations of the parts of a triangle. This is done in the proposed approximate solution. If five-place logarithms are used, this approximate solution is, in general, as accurate numerically as the theoretically more exact one; sometimes this would be true even for six-place logarithms.

5. The summation method is an efficient means of computing r for N, M, and σ may be easily derived from the first, second, and third cumulative summations of a twofold frequency table. ΣXY equals the sum of the second cumulative summation. This method, suggested by Elderton, is more easily applied than the Yule method because it lends itself more readily to computation on an adding machine. A check is determined easily.

6. In the integral

$$w = \int_0^z \frac{dz}{(1 - z^n)^{2/n}},$$

let n be a positive integer greater than 2. Now expand this integral into a series and we get

$$w = z + \frac{2}{n} \cdot \frac{1}{n+1} z^{n+1} + \frac{2}{n} \cdot \frac{n+2}{2n} \cdot \frac{1}{2n+1} z^{2n+1} + \cdots,$$

which series is convergent for $|z| \leqq 1$. In this series let z become the complex variable of the circumference of the unit circle with center the origin; that is, let $z = e^{i\theta}$ and let $w = u + iv$. Then

$$u + iv = e^{i\theta} + \frac{2}{n} \cdot \frac{1}{n+1} e^{(n+1)i\theta} + \frac{2}{n} \cdot \frac{n+2}{2n} \cdot \frac{1}{2n+1} e^{(2n+1)i\theta} + \cdots$$

or

$$u = \cos\theta + \frac{2}{n} \cdot \frac{1}{n+1} \cos(n+1)\theta + \frac{2}{n} \cdot \frac{n+2}{2n} \cdot \frac{1}{2n+1} \cos(2n+1)\theta + \cdots$$

and

$$v = \sin\theta + \frac{2}{n} \cdot \frac{1}{n+1} \sin(n+1)\theta + \frac{2}{n} \cdot \frac{n+2}{2n} \cdot \frac{1}{2n+1} \sin(2n+1)\theta + \cdots.$$

For any assumed value of n as θ varies from 0 to 2π, u and v are the running coördinates of a regular polygon of n sides with the center of the polygon at the origin and with one vertex on the positive axis of u. For n equals 3, we have the equilateral triangle; for n equals 4, the square; and so on for other values of n.

7. Mr. Cary's treatment of right-angled triangles with integral sides was developed from the relation $x + iy = (\xi + i\eta)^2$. The network formed by $\eta = k$, $\xi = k$, $\xi - \eta = k$, $\xi + \eta = k$, ($k = 1, 2, 3$, etc.), corresponds to a network of parabolas in the x, y plane whose equations are

$$y^2 = \pm 2(2k^2)x + (2k^2)^2$$

and

$$x^2 = \pm 2(k^2)y + (k^2)^2.$$

Integral values of x and y lie at the intersections of these parabolas which meet at angles $\theta = n\pi/4$. ($n = 1, 2, 3$.) Solutions obtained by repeated applications of a set of recursion formulæ of the type: $A = a + 2b + 2c$, $B = 2a + b + 2c$, $C = 2a + 2b + 3c$, lie on a parabola through a, b or on $x - y = \pm (a - b)$.

8. Professor Root mentioned briefly the development of the idea of the mathematical laboratory, with particular reference to the courses at the University of Edinburgh and at Massachusetts Institute of Technology. He then indicated the character of the laboratory course as given in the postgraduate school at the Naval Academy, describing briefly some of the exercises given. The laboratory course was shown to differ both in method and subject matter from other courses in mathematics given concurrently. Holding that the laboratory method of instruction is most effective for those topics which require a minimum of explanation and discussion by the teacher and a maximum of performance by the student, he indicated the construction of alignment charts as peculiarly suitable for laboratory exercises.

9. Professor Bramble discussed the possibilities of mathematical laboratory work in which instruments such as planimeters, integrators, integraphs and ellipsographs are used. The exercises as given in this work at the Naval postgraduate school were described. Especial attention was called to an exercise in which the principal axes of inertia of a given area (such as a beam section) are located by means of the Amsler three-wheel integrator.

HARRY ENGLISH, *Secretary-Treasurer.*

TENTH ANNUAL MEETING OF THE OHIO SECTION.

The tenth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, on April 3, 1925, in connection with the meetings of the Ohio College Association and allied societies. Chairman Harris Hancock presided, being relieved by Professor R. B. Wildermuth for an interval.

Forty-seven persons were registered, the following thirty-five being members of the Association:

W. E. Anderson, G. N. Armstrong, C. L. Arnold, Grace Bareis, I. A. Barnett, J. B. Brandeberry, V. B. Caris, E. H. Clarke, Rufus Crane, W. Dancer, O. L.

Dustheimer, T. M. Focke, B. C. Glover, H. Hancock, F. C. Hartwick, E. M. Justin, H. W. Kuhn, C. C. MacDuffee, W. C. McCoy, J. E. Merrill, E. S. Manson, C. N. Moore, Hortense Rickard, W. G. Simon, S. A. Singer, G. W. Spenceley, M. O. Tripp, J. H. Weaver, R. L. Wilder, R. B. Wildermuth, C. O. Williamson, K. D. Swartzel.

At the business session the secretary reported a membership of ninety-seven and eleven institutional members as against eighty-five and nine, respectively, last year. Officers elected for this year are: Chairman, Professor R. D. BOHANNAN, Ohio State University; Secretary-Treasurer, Professor G. N. ARMSTRONG, Ohio Wesleyan University; Third Member of the Executive Committee, Professor C. C. MACDUFFEE, Ohio State University. Professor R. B. Allen, having resigned from the program committee, Professor I. A. Barnett was elected to take his place, and Professor E. H. Clarke was selected to fill the vacancy due to the expiration of Professor C. N. Moore's term. The financial situation of the Section is satisfactory. The committee circulating letters among the high schools of the state on "Why Elect Mathematics" was instructed to continue its work. Professor R. B. Wildermuth was named chairman of the committee to prepare revised letters to be discussed next year.

The Section dinner, with about forty present, served in one of the dining rooms of Ohio Union, was very successful. The evening session, which began with informal after dinner speeches, was held in the same room and was devoted to the last two numbers on the program.

The following nine numbers were presented at the two sessions:

(1) Chairman's Address: "A notion which includes that of divisibility" by Professor HARRIS HANCOCK, University of Cincinnati.

(2) "Some criteria for the rejection of doubtful data" by Professor W. E. ANDERSON, Miami University.

(3) "College algebra for freshmen" by Professor E. H. CLARKE, Hiram College.

(4) "On the definition of a continuous curve" by Professor R. L. WILDER, Ohio State University.

(5) "Some new examples of modular non-euclidean geometries and a tactical method for their discovery" by Professor C. L. ARNOLD, Ohio State University.

(6) "Some new methods for finding the roots of algebraic equations" by Professor C. W. SPENCELEY, Miami University.

(7) "Mathematics as related to the American college" by Professor O. L. DUSTHEIMER, Baldwin-Wallace College.

(8) "Side-lights on the international congress at Toronto" by various Ohio mathematicians who attended the congress.

(9) "Some statistical data relative to the lack of preparation of college freshmen in mathematics"; discussion opened by Professor HARRIS HANCOCK, University of Cincinnati.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. In the beginning Professor Hancock gave definitions of *realms of rationality, algebraic numbers, integers* and *units*. Then it was shown by the well-known example that 21 in the realm $R(\sqrt{-5})$ may be factored in several different ways. Kummer numbers were introduced and their connection with the Dedekind ideals was indicated. By the introduction of the prime ideals a unique factorization in the extended realms may be effected. A number λ is divisible by the complex $[\alpha, \beta]$ if λ is contained in this complex, that is, if integers ξ and η may be found such that $\lambda = \alpha\xi + \beta\eta$. This notion, superfluous in the usual realm of natural (usual) numbers, is necessary in the more extended realms.

2. The criteria of Peirce, Stone, Chauvenet, and the U. S. Coast and Geodetic Survey were presented by Professor Anderson, especial attention being given to the development and application of Peirce's criterion. Criticisms by Airy, Stone, Winlock, and Glaisher were given. The present practice of the U. S. Naval Observatory regarding rejection of doubtful observations was given. The object of the paper was to try to show the best and most approved methods of handling this problem which has engaged the attention particularly of those who have been concerned with the adjustment of observations, including data of more or less doubtful value.

3. Professor Clarke pointed out the lack of well-defined objectives in our present courses in algebra for college freshmen. He suggested as essential aims: (1) the development of the idea of functional relationship, (2) clear-headed thinking on the equation, (3) accuracy in numerical work.

4. In Professor Wilder's paper there was presented a resumé of some of the criteria developed in recent years in the field of analysis situs for determining whether given point-sets fall in the class of curves which can be defined by continuous functions $x = f_1(t)$, $y = f_2(t)$, $[0 \leqq t \leqq 1]$. The following simple test, similar to that given by Schoenflies for plane continuous curves, was introduced: M being a plane continuum, and K any circle, denote by N the set of points of M interior to K and by T the set $K + N$. Then in order that M should be a continuous curve, it is necessary and sufficient that for every such set T and every positive number E the number of domains complementary to T and of diameter greater than E should be finite.

5. Professor Arnold noted that the finite modular geometries (without parallels) have $n(n-1)+1$ elements, where "n" is the number of points on a line. It was shown that the set of *differences* between the numbers of any line may be used to assure the discovery of all the possible geometries for any given value of n. Geometries exist for $n = 2, 3, 4, 5, 6, 8$, and 9, also probably for many greater values of n. The number of known geometries is: For $n = 2, 3, 5$, one each; for $n = 4$, two; for $n = 6$, five; for $n = 7$, none; for $n = 8$, six; for $n = 9$, at least one. Those for n greater than 5 are believed to be new.

6. In a polynomial $p_0x^n + p_1x^{n-1} + p_2x^{n-2} \cdots + p_n = 0$, Professor Spenceley pointed out that a positive real root x, which is large in comparison with S, the sum of all the other roots, is approximately given by $-p_1/p_0$. Any polynomial whatever may be transformed into one whose roots are of this peculiar

type by several repetitions of the transformation

$$x = \frac{r^2}{y - b} + a,$$

which is geometrically equivalent to reflecting the roots of the original equation in a circle in such a way as to make the desired root go over into a large root while the sum of the remaining roots is made small.

7. Professor Dustheimer had 300 catalogues examined from every state in the U. S. and every province of Canada. 50% require 2 units of secondary mathematics for entrance. 50% require from 18 to 24 hours for a mathematics major. 50% require 6 hours or less of mathematics for graduation. 30% offer from 20 to 30 hours of mathematics. 80% have no connection between mathematics and physics. 46% offer from 2 to 6 hours of astronomy; 30% have some connection between mathematics and astronomy; 30% teach no astronomy.

Answers to the questionnaire from 55 colleges (33 Ohio colleges) show the following: Only 16% require students majoring in mathematics to take history of mathematics. Sixty-one per cent. give courses in the teaching of mathematics while very few give courses combining the history and teaching of mathematics. Many of these courses are lecture courses, but nearly all of them use a textbook. The courses average 2 hours per semester and are generally open to juniors and seniors.

8. These reports were informal and humorous rather than scientific, dealing mainly with personal experiences.

9. Professor Hancock had Professor Barnett report the result of identical tests given to college freshmen who had had some college mathematics in the University of Cincinnati, Ohio State University and Adelbert College. The results in the three institutions were in fair agreement but decidedly unsatisfactory although the tests were extremely easy. Professor Swartzel explained a plan in use at the University of Pittsburgh whereby they demote students until they find their true level of preparation. Few failures need be recorded by this device, which works beneficially. Professor Holl reported the method used this year in Ohio Wesleyan University in sectioning freshmen entering with only one year of algebra. The discussion extended itself to nearly all phases of present-day education.

Professor Barnett presented the matter of the Carus monographs in an effective talk.

G. N. ARMSTRONG, *Secretary-Treasurer.*

THE ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION.

The ninth annual meeting of the Rocky Mountain Section was held at the University of Wyoming, Laramie, on April 10 and 11. There were thirty-six present, including the following ten members of the Association: I. M. DeLong,

Philip Fitch, J. C. Fitterer, H. C. Gossard, Claribel Kendall, G. H. Light, S. L. Macdonald, Letitia Odell, O. H. Rechard and H. E. Russell.

The section voted to hold the next meeting at the Colorado Agricultural College. The following officers were elected: S. L. MACDONALD, chairman; W. V. LOVITT, vice-chairman; PHILIP FITCH, secretary; G. H. LIGHT, treasurer.

Three committees were appointed, one to draft resolutions on the death of Dr. Carl E. Stromquist, formerly head of the department of mathematics at the University of Wyoming and a member of this section; one to formulate plans for the use of standard tests in connection with the teaching of mathematics in the colleges; and one to consider the advisability of procuring a speaker from outside the section for the next meeting.

On Friday evening, the members of the section were guests at a dinner given by the home economics department of the University. President A. C. Crane delivered an address of welcome on this occasion and expressed his pleasure at having the section meeting held at his institution. Professor S. L. Macdonald responded in a fitting manner in behalf of the guests.

Later in the evening the women members of the section were entertained at a production given by the Coffer-Miller players.

The following nine papers were read:

(1) "An endowment for the publication of the results of mathematical research" by Professor I. M. DELONG.

(2) "Mathematics as an aid in agricultural experimentation" by Professor A. G. CLARK (by invitation).

(3) "The relation of standard tests to the teaching of collegiate mathematics" by Professor H. C. GOSSARD.

(4) "Baade's asteroid" by Dean H. A. HOWE.

(5) "Points of view on the multiplicative axiom" by Professor C. H. RECHARD.

(6) "On the quinquenary cubic expressible as the sum of seven cubes" by Professor C. H. SISAM.

(7) "Integration in series" by Professor G. H. LIGHT.

(8) "Analysis of certain types of composite curves" by Mr. PHILIP FITCH.

(9) "On a tetrahedron" by Professor H. C. GOSSARD.

In the absence of the authors, the paper by Dean Howe was read by Professor Russell and the one by Professor Sisam by title only.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor DeLong pointed out the urgent need of funds for publishing results of mathematical research and made an appeal to all interested in mathematics to assist the American Mathematical Society in establishing an endowment large enough to insure the publication of the fine results that are being achieved by mathematicians in this country.

2. Professor Clark mentioned, in a brief way, various typical problems arising in agricultural experimental work where the need for mathematical treatment

was obvious. An actual problem, dealing with the peculiar results of the cross of two varieties of barley, was solved.

3. This paper was a report of experiments with speed, accuracy, and power tests in freshman and sophomore college mathematics. As a result of this report, the colleges and universities of the Rocky Mountain Section of the Association voted to coöperate in a continuation of this experiment.

4. The discovery of Baade's asteroid last October was made by photography. The computation of a preliminary orbit at the Student's Observatory of the University of California was beset with special difficulties, but when a reasonably correct orbit was finally obtained, a request from Denver brought an ephemeris of the planet, furnished in advance of publication, so that measures of its position with the 20-inch Denver telescope might begin at once. The orbit has been found to be the most eccentric planetary orbit known, with one exception. The inclination of the orbit is among the dozen highest. Only three planetoids come nearer to the earth than it does. Furthermore it is unusual in that it varies regularly in brightness in a period of a few hours. This is probably an indication of axial rotation, perhaps the first shown by any planetoid. Despite the faintness of this planet, a very large number of observations has been made at Denver, but the approaching opposition of the asteroid will cause a termination of the series.

5. Professor Rechard discussed the multiplicative axiom as brought to the fore by Zermelo's Theorem. The discussion centered around the points of view of various mathematicians, especially Baire, Borel, Hadamard, and Lebesgue. A summary of the principal problems and special fields affected by one's point of view on the axiom was included.

7. This paper gives methods for finding the solution of the differential equation

$$(x - a)^2 p_0 \frac{d^2y}{dx^2} + (x - a) p_1 \frac{dy}{dx} + p_2 y = X.$$

It is shown when this equation will have a solution in ascending and descending powers of $(x - a)$ as well as when a logarithmic solution occurs. The particular integral is found in a very simple manner.

8. Mr. Fitch discussed the analysis of composite curves obtained from experimental data arising from observations made on thermoluminescent and similar effects. He pointed out the necessity of being guided by the scientific facts involved while attempting such an analysis.

9. Following a suggestion by Dr. Morley of Johns Hopkins the author of this paper presented twenty-one relations between the edges and faces of a tetrahedron. The equation of the absolute is expressed first so that its coefficients are in terms of the edges of the tetrahedron and second in terms of its faces. The twenty-one resulting coefficients are then equated giving the desired relations.

PHILIP FITCH, *Secretary.*

OUTLINES OF RESEARCH: GENERAL ANALYSIS.[1]

By T. H. HILDEBRANDT, University of Michigan.

I feel that there is some need of apology that I should be the one to speak on the subject indicated. In thinking about the matter, it seemed to me that there were others who were far better suited to the task, others who had had considerable experience in directing research in general analysis or who had made rather significant contributions to the subject. This feeling was quite strong when the chairman of your program committee wrote to me about the matter of giving this paper at this time, and I tried to sidestep the issue. But he was insistent, and the general manager of the Association contributed his share towards spurring me on, so that I reluctantly acquiesced. If the results are not a credit to the Association or to the subject, the responsibility is theirs.

In a subject developing so rapidly as general analysis, it is to be expected that some of the things which I suggest as possibilities have been already proved, or disproved, and it may be that some have been published and have escaped my attention. I offer proper apologies for this contingency at this point.

There is no need of offering apologies for the subject of general analysis. By this term I mean to cover that body of mathematical science which in recent years has devoted itself to the task of replacing particular instances by general inclusive ones, especially in the field of analysis. The habit of generalization is one that is inculcated into the budding mathematician from the time when he begins the study of elementary algebra. The training continues, and for most of us the habit thus formed never ceases. Any one who has attempted generalization in any field can easily subscribe to the fact that such generalization has resulted, on the one hand, in a sharper analysis of the essentials of the theory, *i.e.*, in separating out the items peculiar to the special situation from those which are of general import; secondly, to a simplification; and lastly, to a widening of the sphere of influence and inclusion of greater results. General analysis has done this very thing for the realms into which it has entered up to the present time, *viz.*, the theory of abstract sets, theory of functions on abstract sets, and theory of functionals or transformations. It has abundantly verified the truth of the tenet of faith of general analysis, so ably phrased by one of the leaders in this field, E. H. Moore:[2]

The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to those central features.

In this paper, it is obviously impossible to give what would be a complete outline of research in a field which has extended its scope and developed so rapidly in the two decades of its existence. As a matter of fact the word "outline" in the title is rather a misnomer. A more appropriate title would be

[1] Read before the Mathematical Association of America at Washington, D. C., on December 31, 1924.

[2] *New Haven Colloquium Lectures* (1910), p. 1.

"Some Suggestions for Research in General Analysis." Naturally a selection is rather a difficult matter. I have confined myself to a few indications in two fields: I. Theory of Abstract Sets, and II. Functionals, particularly Linear Functionals. Most of the suggestions made are in the nature of questions which have arisen in the course of reading. We might well say that "the beginning of research is the question and the question mark," and that the person who has formed the habit of being profuse with questions in his reading is on the high road to research.

I. Theory of Abstract Sets. Before treating questions which relate to the theory of abstract sets, it may be desirable to give a bird's-eye view of the theory in its historical development.

We postulate in such a theory a set of general elements, the class $\mathfrak{P}$ of elements p.[1] We can assume that the class $\mathfrak{P}$ is left entirely unconditioned, *i.e.*, the elements of the class are general. In that case, in order to be able to say anything, we must shift the responsibility to other quarters. The usual method is to assume the existence of functions or rather a class of functions on $\mathfrak{P}$ with real or complex values, and study the interrelations between the functions, as being properties of the class of functions. This was the point of view of E. H. Moore in his general analysis, whose primary interest lay in the building of a foundation for the generalization of the theory of linear integral equations, and later, Hellinger integrals. We shall not discuss this point of view. The other method of approach is to study interrelations as existing between the elements of the class $\mathfrak{P}$, *i.e.*, condition the class. This is the point of view of Fréchet as exposed first in his thesis, "Sur quelques points du Calcul Fonctionnel,"[2] whose purpose was to generalize the existing theory of point-sets.

Now, in the theory of point-sets a central rôle is played by the notion of *limiting element*. Fréchet approached this notion *via* the notion of *limit of a sequence*, and hence postulated in the first place that in the general class $\mathfrak{P}$ there was defined the limit of a sequence, this notion being subject to the following three conditions: (*a*) the limit of a sequence is unique, (*b*) a sequence which consists of one element repeated has this element as limit, (*c*) any subsequence of a sequence having a limit has the same element as limit. Such a class he calls a class $\mathfrak{L}$. A limiting element of a subclass of a class $\mathfrak{L}$ is defined as the limit of a sequence of distinct elements chosen from the subclass. A class $\mathfrak{L}$ suffices as a basis for the discussion of some continuity properties of functions,[3] but does not carry very far in the theory of abstract sets. For the latter he assumes that between pairs of elements of the class $\mathfrak{P}$ [4] there is defined a function

[1] We follow E. H. Moore in notation, for the simple reason that he has made a good advance towards the question of uniformising notation in this field, *viz.*: small letters for elements, capital letters for transformations, Greek letters for functions, and capital German letters for classes. The use of the letter p is due to the fact that it does not have a preconceived connotation.

[2] *Rendiconti di Palermo*, vol. XXII (1906), pp. 1–74.

[3] Continuity properties of functions seem to demand very little, as indicated for instance by Fréchet: *Ann. de l'Éc. Norm. Sup.*, ser. 3, vol. 37 (1920–1), pp. 356 ff.

[4] Fréchet assumes that this is true for *all* pairs of elements, but this is not necessary. *Cf.* Gross: *Sitzungsberichte der Wiener Akademie*, vol. 123, Abt. 2a, pp. 802 ff.

$\delta(p_1p_2)$, subject to the conditions: (1) $\delta(p_1p_2) \geqq 0$; (2) If $\delta(p_1p_2)$ exists, then $\delta(p_2p_1)$ exists and they are equal; (3) $\delta(p_1p_2) = 0$ if and only if $p_1 = p_2$; (4) If $\delta(p_1p_2)$ and $\delta(p_2p_3)$ exist, then $\delta(p_1p_3)$ exists and

$$\delta(p_1p_3) \leqq \delta(p_1p_2) + \delta(p_2p_3).$$

Limit in such a space is defined as being equivalent to

$$\lim_{n \to \infty} \delta(p_np) = 0.$$

Such a space is generally called a metric space, and is denoted by the symbol $\mathfrak{D}$. It forms a natural and elegant basis for a general theory of point-sets in the abstract field, as has been illustrated for instance by Hahn in his *Theorie der reellen Funktionen*.

The principal later advance in the foundations of the theory has come through the realization that *limiting element* of a set of elements is the central figure in a theory of abstract sets, and need not be approached through the limit of a sequence, but can be postulated directly. This was pointed out by Riesz in his paper "Stetigskeitbegriff und abstrakte Mengenlehre" before the Mathematical Congress in Rome in 1908. Limiting element is subjected by him to the following postulates: (1) if p is a limiting element of a subclass $\mathfrak{P}_0$ of $\mathfrak{P}$, then it is also a limiting element of every subclass of $\mathfrak{P}$ containing $\mathfrak{P}_0$, (2) if p is a limiting element of a subclass $\mathfrak{P}_0$ of $\mathfrak{P}$, and $\mathfrak{P}_0$ is divided into two classes, then it is also a limiting element of one of these two classes, (3) a class consisting of a single element has no limiting element, and (4) limiting element of a subclass of $\mathfrak{P}$ is determined uniquely by the totality of classes of which it is a limiting element.

Starting from these ideas on limiting elements, R. E. Root [1] was led to connect the idea of limiting element with a special group of subclasses of the fundamental class usually called vicinities. The same idea was obtained independently by Hausdorff [2] in 1914 and again by Fréchet [3] in 1918. The latter considers what properties of a limiting element are necessary and sufficient for the existence of a vicinity notion. He finds that the notion of vicinity with limiting element defined, namely, p is a limiting element of a subclass $\mathfrak{P}_0$ of $\mathfrak{P}$, if every vicinity of p contains an element of $\mathfrak{P}_0$ different from p, is equivalent to the condition (1) of Riesz, together with the fact that whether an element p of $\mathfrak{P}$ is or is not a limiting element of a subclass depends on the elements of the subclass other than p. These conditions are very slight, and it seems that the future will

[1] "Limits in Terms of Order," *Trans. Amer. Math. Soc.*, vol. 15 (1914), pp. 51–71. Although Root bases his discussion fundamentally on a notion of betweenness, *i.e.*, of order, he uses this notion only to define the set of special subclasses in terms of which all later reasoning is done. It might be interesting to study limiting element directly in terms of order without the intervention of the vicinity notion, and determine what properties of order produce for instance the Riesz properties of limiting element.

[2] *Grundzuege der Mengenlehre*, pp. 212 ff.

[3] *Bull. des Sci. Math.*, 53 (1918), pp. 138–156.

probably link up the theory of abstract sets with classes in which a vicinity is defined (denoted by $\mathfrak{B}$), properly conditioned.[1]

We note finally some work on a general limit due to E. H. Moore and H. L. Smith[2] the prime object of which is to obtain a general limit to include the kind of limits which occur in connection with the definitions of integration of the Riemann type. They postulate an order defined by a relation R. This order is subject to two conditions: (1) transitivity: if p_1Rp_2 and p_2Rp_3, then p_1Rp_3, and (2) for every p_1 and p_2 there exists a p_3 such that p_3Rp_1 and p_3Rp_2. Viewed from the point of view of abstract sets[3] this naturally is a special instance in the theory given by Root which shows that any order based on a betweenness relation (obviously resultant from the transitivity postulate in this case) gives rise to a vicinity notion.

We come now to the questions which might be suggested with respect to the existing theory. In the theory of linear point sets, the difference between the notion of limit and that of limiting element is first that limit is connected with a sequence, *i.e.*, an ordered set, while limiting element is associated with a set of elements, order not prerequisite, and secondly that if a sequence has a limit, it has only one, while limiting elements of a set may be many. So far we have abstracted the notion of limit by preserving the sequence idea. It might be interesting to study what kind of a theory would result if we assumed as fundamental the notion of limit as attached to a class (ordered or not) emphasizing the uniqueness feature rather than the sequential idea.[4] From such a limit suitably conditioned we might get then notions of limiting elements. If the notion of order is to reign supreme in the notion of limit, it might be interesting to study what happens if limit is attached to sets which are ordered, the sets being no longer denumerable, necessarily, the order being for instance of the monotonic type where of two elements one precedes another, or of the type suggested by Moore and Smith, where every two elements have a common successor. One of the points of separation between limit and limiting element is that limiting element need not attach itself to a denumerable set.[5] Perhaps by starting with denumerability thrown into the discard, but with unique element and order present, we may be able to get a limit which will give rise to as general a limiting element as can be obtained by direct postulation.

Another thing which might well be done is to clear up the relationships which exist between different definitions, in the theory of abstract sets. For instance take the definition of limiting elements of different power or order. In Fréchet's thesis we find, besides the concept of limiting element, the notion of element of condensation—a limiting element being an element of condensation

[1] This sentence is the burden of the remarks made by E. W. Chittenden in discussing this paper.

[2] *Cf. Amer. Jour. of Math.*, 44 (1922), pp. 102–121.

[3] Moore and Smith are interested mainly in limit as applied to functions, of the type: limit as n increases indefinitely.

[4] *Cf.* Hausdorff, *loc. cit.*, pp. 232 ff.

[5] *Cf.* Root, *loc. cit.*, pp. 67 ff.

of a class, if it remains a limiting element of every subclass obtainable by removing a denumerable set of elements. Then in Hausdorff [1] we have a limiting element of order α in a class of type $\mathfrak{B}$, if every vicinity of p contains one element of the class (not necessarily different from p); a limiting element of order β if every vicinity of p contains a denumerable infinity of elements, and of order γ if every vicinity of p contains a nondenumerable infinity. Along the same line one can define a limiting element of *order* $\mathfrak{m}$ if every vicinity of p contains a set of elements of power $\mathfrak{m}$, but not every vicinity contains a set of power greater than $\mathfrak{m}$. Kuratowski and Sierpinski [2] have defined a limiting element which involves the notion of interiority (p is interior to $\mathfrak{P}_0$ if it belongs to $\mathfrak{P}_0$ and is not a limiting element of any set consisting entirely of elements not belonging to $\mathfrak{P}_0$) which might perhaps be called limiting elements of *power* $\mathfrak{m}$: p is a limiting element of $\mathfrak{P}_0$ of power $\mathfrak{m}$ if every vicinity of p has interior to it a set of elements of $\mathfrak{P}_0$ of power $\mathfrak{m}$, but not every vicinity contains in the interior sense a set of power greater than $\mathfrak{m}$.[3] It would obviously be desirable to find out to what extent these various concepts overlap, to determine existing relationships and to point out which had best be discarded and which used.

Closely connected with these different kinds of limiting elements one finds properties of classes which are for instance generalizations of the Weierstrass-Bolzano theorem. In Fréchet we find compactness, in which every infinite subclass has a limiting element, and condensed, in which every nondenumerably infinite subclass has an element of condensation. Then in Gross [4] we find compactness of different orders: a class being $\mathfrak{m}$-compact if every subclass of order greater than $\mathfrak{m}$ has a limiting element. Obviously compactness of various kinds could be combined with limiting elements of different orders, thus arriving at varied definitions. The interrelations and applicability of these ideas is desired. Among these things it might perhaps be possible to obtain theorems similar to the one due to Gross and Fréchet [5] that in a class $\mathfrak{D}$ the notions condensed and separable (a class is separable when it belongs to the derived class of a denumerable set) are equivalent. In this connection too one ought not to overlook the notion of perfectly compact due to R. L. Moore,[6] and defined in terms of a monotonic set of classes, *viz.*, that every monotonic set of subclasses of $\mathfrak{P}_0$ has a common element or common limiting element.[7]

Of the theorems which have engaged the attention of workers in the theory of abstract sets, the Borel-Lebesgue theorem has no doubt claimed the maximum

[1] *Loc. cit.*, pp. 219 ff.

[2] *Fundamenta Mathematicæ*, vol. II (1921), p. 173.

[3] In an article by E. W. Chittenden, *Bull. of Amer. Math. Soc.*, vol. 30 (1924), pp. 512, 3, which has appeared since this paper was read, limiting elements of order and power $\mathfrak{m}$ are used and called nuclear and hypernuclear points respectively.

[4] *Loc. cit.*, p. 805.

[5] Gross, *loc. cit.*, p. 806. Fréchet, *Ann. de l'Ec. Norm. Sup.*, ser. 3, vol. 37 (1920–1), pp. 349 ff.

[6] *Proceedings of the Nat. Acad. of Sci.*, 5 (1919), pp. 206–10.

[7] In the article by Chittenden referred to above, it is shown that the notion of perfect compactness is equivalent to a kind of complete compactness based on limiting elements of various orders, thus justifying the name perfectly compact.

interest. The theorem of Borel-Lebesgue states that under suitable conditions a set of subclasses, such that each member of a given class is interior to some one of the subclasses of this set, can be replaced by a finite number of classes chosen from this set, and having the same property. Fréchet's proof of this theorem involved the class $\mathfrak{D}$; later it was extended to classes of the type $\mathfrak{L}$ and $\mathfrak{B}$. In the proofs of the latter [1] there intervene properties of well-ordered sets of the transfinite variety, *i.e.*, the Zermelo axiom and theorem is used. In consideration of other cases in which the theory of transfinites has been used in analysis, and replaced by other reasoning not involving them, it might naturally be asked whether such is possible with respect to the proofs of the Borel-Lebesgue theorem in a class $\mathfrak{B}$ for instance.

Another direction does not seem to have been pursued. The theorem of Borel-Lebesgue abstractly considered is concerned with the possibility of replacing a set of subclasses having a certain property by a finite (or denumerable?) set having the same property. In the theorems so far considered, the property in question is that every element of a given set shall be interior to a given subclass. The question may be suggested whether it is possible to derive theorems of the Borel-Lebesgue type, where this property is replaced by some other. In the theory of measure one finds somewhat similar theorems, and there may be some of similar character in general point-set theory.

II. Functionals. In discussing functionals, *i.e.*, functions of functions, it is possible to proceed in many different ways. We may follow E. H. Moore in his first general analysis,[2] starting with a general class $\mathfrak{P}$, a class of complex-valued functions on $\mathfrak{P}$, subject to definite properties, and a linear operator with postulated properties; or follow him in his later work in which the linear operator is obtained constructively and the properties of the general class of functions derived therefrom. We shall instead follow a trend which seems to be indicated in the recent developments of analysis, Lebesgue integration, Stieltjes integration and kindred operators. We proceed by postulating a class $\mathfrak{P}$ of elements p, general and unconditioned; a class $\mathfrak{M}$ of functions μ, complex-valued, on the range $\mathfrak{P}$. This class $\mathfrak{M}$ is assumed to be linear, *i.e.*, if μ_1 and μ_2 are two functions of $\mathfrak{M}$ and a_1 and a_2 any complex numbers, then $a_1\mu_1 + a_2\mu_2$ also belongs to $\mathfrak{M}$. On $\mathfrak{M}$ we assume the existence of a norm or modulus,[3] $N\mu$, with positive or zero values and with the properties:

$$(1)\quad N(\mu_1 + \mu_2) \leqq N\mu_1 + N\mu_2,$$

$$(2)\quad N(a\mu) = |a| N\mu.$$

We assume that any function μ such that $N\mu = 0$ is equivalent to the function which is identically zero on $\mathfrak{P}$, and as a consequence two functions, the norm of

[1] *Cf.* R. L. Moore, *loc. cit.;* Fréchet, *Ann. de l'Ec.*, article pp. 348–9; Kuratowski and Sierpinski, *loc. cit.*, and Chittenden, *loc. cit.*

[2] *New Haven Colloquium Lectures*, 1910.

[3] This norm $N\mu$ usually appears symbolized $\| \mu \|$.

whose difference is zero, are considered as being identical. With respect to the norm N we assume that the class $\mathfrak{M}$ is complete or satisfies a Cauchy condition, *i.e.*, if the sequence μ_n is such that

$$\lim_{n,\,m} N(\mu_n - \mu_m) = 0,$$

then there exists a μ of class $\mathfrak{M}$ such that

$$\lim_{n} N(\mu_n - \mu) = 0.$$

Examples of classes $\mathfrak{M}$ and norms are plentiful. *E.g.*, if the class $\mathfrak{P}$ is the class of integers 1, 2, 3, $\cdots$, which we shall call $\mathfrak{P}^{\text{III}}$ in the sequel, then $\mathfrak{M}$ is some class of sequences. If $\mathfrak{M}$ is the class of sequences such that $\Sigma_p|\mu_p|^2$ is convergent, then $N\mu = \sqrt{\Sigma_p|\mu_p|^2}$; or if $\mathfrak{M}$ is the class such that $\Sigma|\mu_p - \mu_{p+1}|$ is convergent, then $N\mu = \Sigma|\mu_p - \mu_{p+1}| + |\mu_1|$; and so on. Or if $\mathfrak{P}$ is the linear interval $0 \leqq p \leqq 1$, which we shall call $\mathfrak{P}^{\text{IV}}$, then $\mathfrak{M}$ is a class of functions on the real interval; *e.g.*, the class of continuous functions with $N\mu =$ maximum of the $|\mu(p)|$ for p on $\mathfrak{P}^{\text{IV}}$; or if $\mathfrak{M}$ is the class of functions of bounded variation, then $N\mu$ is the total variation of μ; and so on.[1]

A class $\mathfrak{M}$ of the type discussed here is evidently of the character of the class $\mathfrak{D}$ of Fréchet with $\delta(\mu_1, \mu_2) = N(\mu_1 - \mu_2)$, provided $N\mu = 0$ is equivalent to μ identically zero.

This basis can be replaced by what seems to be a more general basis, called a *vectorial space*.[2] In this type of space, the function μ on $\mathfrak{P}$, complex-valued, is replaced by a *vector*, a suitably conditioned element. With respect to the class of these vectors, it is assumed that there is defined an addition process which is commutative and associative; and multiplication by real (or complex) numbers, which is commutative, associative, and distributive with respect to addition. Finally there is assumed a norm or modulus (usually denoted $\|\ \|$) which transforms the elements into real non-negative numbers, with properties similar to those given above.

This suggests a problem, which might be investigated, *viz.*, whether a vectorial space of this kind is in fact more general than a function space on a general range. Most of the examples of vectorial spaces so far considered can be thought of as complex-valued functions on a certain range. If it could be shown that every vectorial space can so be expressed (the converse is obvious), there would be a considerable gain in simplicity.

A question might be asked with respect to the norm as postulated above, especially as regards the condition (2):

$$Na\mu = |a|N\mu.$$

[1] A large number of interesting examples of this kind are discussed by Hahn, *Monatshefte f. Math. u. Physik*, vol. 32 (1922), pp. 1 ff.

[2] *Cf.* Banach, *Fund. Math.*, 2 (1921), pp. 134 ff.; Wiener, *Bull. de la Soc. Math. de France*, 50 (1922), pp. 123 ff.

Fréchet [1] has shown that in the class of all sequences, with limit defined

$$\lim_n \mu_n = \mu \text{ if and only if } \lim_n \mu_n(p) = \mu(p) \text{ for every } p,$$

it is possible to find a distance function giving an equivalent definition of limit. However this distance function considered as a norm does not satisfy this condition (2), and the question is whether it is possible to find such a distance function in this case. A similar question might be asked with respect to functions on the infinite interval.[2] On the other hand, it may be possible to obtain certain results in this theory, especially those relative to linear functionals which seem to require this postulate on norm, without this postulate, or by replacing it with some weaker postulate.[3]

Along the same lines comes the question: Is there for each class of functions a "most natural norm"?[4] This question is of course indefinite because of the indefiniteness of the term "most natural." Without attempting to make it precise, perhaps a few examples will make my meaning clear. In the class of continuous functions on $\mathfrak{P}^{IV}$ the most natural norm seems to be the maximum of the absolute value of the function, *i.e.*, connected with uniform convergence; in the class of Lebesgue integrable functions, the most natural norm seems to be the Lebesgue integral of the absolute value of the function; in the class of functions which are measurable and bounded, if a set of measure zero be neglected, the most natural norm seems to be the least upper bound of the absolute value of the function, obtainable by neglecting sets of measure zero; for functions of Lebesgue integrable square it seems to be the square root of the integral of the square; for functions defined on $\mathfrak{P}^{III}$, for the class of absolutely convergent series, $N\mu = \Sigma|\mu|$, for the class of convergent series $N\mu = \max |\Sigma_1^n \mu|$ and so on.[5]

For the sequel we shall assume that we are working in a general function space with a norm satisfying the conditions postulated at the beginning of this section.

We take up first some problems with respect to linear modular operations. We shall say that U is an *operation* on $\mathfrak{M}$ if it transforms every function μ of $\mathfrak{M}$ into a real or complex number; *linear* if for μ_1 and μ_2 of $\mathfrak{M}$, and a_1 and a_2 complex numbers,

$$U(a_1\mu_1 + a_2\mu_2) = a_1U(\mu_1) + a_2U(\mu_2);$$

modular if there exists a number M such that for every μ,

$$|U(\mu)| \leqq M \cdot N\mu.$$

In most of the current theories, this type of linear modular operator has a

[1] Thesis (*Pal. Rend.*, vol. 22), p. 40.
[2] *Cf.* Gateaux, *Atti di Lincei, Rend.*, ser. 5, v. 23^2 (1914), p. 310.
[3] *Cf., e.g.*, Wiener, *loc. cit.*, p. 124.
[4] *Cf.* in this connection Levy, *Analyse Fonctionelle*, pp. 17 ff.
[5] Further illustrations may be found in the article by Hahn referred to above.

kind of bilinear character. For instance if $\mathfrak{M}$ is class of continuous functions on $\mathfrak{P}^{IV}$, then there exists a function of bounded variation α such that

$$U(\mu) = \int \mu d\alpha,$$

the integral being a Stieltjes integral; if $\mathfrak{M}$ is the class of functions of integrable square on $\mathfrak{P}^{IV}$, then there exists a function of integrable square α such that

$$U(\mu) = \int \alpha\mu,$$

the integral being taken in the Lebesgue sense. If $\mathfrak{M}$ is the class of absolutely convergent series on $\mathfrak{P}^{III}$, then there exists a bounded sequence α such that

$$U(\mu) = \Sigma\alpha\mu$$

and so on. This naturally suggests the question under what conditions on N and the class $\mathfrak{M}$ does there exist a bilinear operator and a function μ' of a class $\mathfrak{M}'$ preferably of the same type as $\mathfrak{M}$ such that a linear modular operator on $\mathfrak{M}$ has the form

$$U(\mu) = J\mu'\mu?$$

If it were possible to derive such a theorem, it would not be difficult to consider such questions as biorthogonality (orthogonality of functions seems to be a property confined to functions where $\mathfrak{M}'$ and $\mathfrak{M}$ are the same class, which in ordinary analysis is true on $\mathfrak{P}^{IV}$ only (?) for functions of Lebesgue integrable square, and on $\mathfrak{P}^{III}$ for sequences of convergent square) as between functions of two associated classes.[1]

We come next to the linear modular functional transformations or functionals. T is said to be a functional transformation on $\mathfrak{M}'$ to $\mathfrak{M}''$ if it transforms every function μ' of $\mathfrak{M}'$ into a function of the class $\mathfrak{M}''$, it being assumed that the two classes in question are of the type of the class $\mathfrak{M}$ which we are considering. Linearity and modular properties are similar to the case of operations, which are special functional transformations in which the class $\mathfrak{M}''$ is a class of complex numbers, $\mathfrak{P}$ consisting of a single element. Probably the most interesting subject in functional transformations is the question of finding inverses, *i.e.*, solving functional equations of the form $T\mu' = \mu''$. Linear integral equations are a special case of this situation with the class $\mathfrak{M}'$ the same as $\mathfrak{M}''$. Some work in connection with the case in which $\mathfrak{M}'$ and $\mathfrak{M}''$ are the same and with respect to so-called completely continuous transformations has been done by Riesz.[2] It is a question whether his methods cannot perhaps be extended to other kinds of transformations. Perhaps the methods discussed in the valuable manual by Riesz on infinitely many variables[3] can be carried over or adapted to the func-

[1] This bilinearity feature has attracted attention previously. *Cf.* the work of E. H. Moore on Integral Equations in General Analysis: *Bull. Amer. Soc.*, vol. 17 (1912), pp. 348 ff. See also the article by Hahn previously mentioned.

[2] *Cf. Acta Mathematica*, vol. 41 (1918), pp. 71–98.

[3] *Equations Linéaires à une Infinité d'Inconnues*, Paris, 1913.

tional field. It might be desirable to do so. Possibly too this problem can be discussed by reducing it to a case of solution of equations in infinitely many variables, and then applying existent results in this latter theory.[1] The usual mode of procedure, especially when dealing with functions on the linear interval, is to perform a transformation *via* a complete system of normed orthogonal functions, which establishes a one-to-one correspondence between a sequence and a function on $\mathfrak{P}^{IV}$. In the general theory under discussion, however, we are not equipped with orthogonal, or even with biorthogonal, systems of functions. A method of attacking this latter problem would be to utilize some of the properties of such a system of functions. Some which might be possible of generalization are the following: If the sequence μ_n is a system of Lebesgue integrable square functions on $\mathfrak{P}^{IV}$, which is normed and orthogonal, then the coefficients in the expansion of any function whose square is Lebesgue integrable in a series of the μ_n can be determined by finding for each n the constants $a_1, \cdots a_n$ such that

$$\int_0^1 (\mu - \Sigma_1^n a_m \mu_m)^2$$

is a minimum. If we extract the square root of this integral, then we have here a case of a norm in this class of functions. These a_m are unique for a given set of linearly independent functions. The other property is that the a_m do not change with n. If the set μ_n is complete, then the minimum value of the integral approaches zero with n.

The first idea is easily generalized. Assume a sequence of linearly independent functions μ_n of the class $\mathfrak{M}$. Then for every function μ there exist constants a_{mn} such that

$$N(\mu - \Sigma_1^n a_{mn} \mu_m)$$

is a minimum. These constants are not necessarily unique. They will be unique if for instance the norm has the property that the equality

$$N(\mu_1 + \mu_2) = N\mu_1 + N\mu_2$$

holds if and only if there exists a real positive constant a such that $\mu_1 = a\mu_2$. Whether the second property that the a_{mn} are independent of n can be generalized is a question. Perhaps it may be possible to show that there exist linear combinations of the given functions, which are linearly equivalent to the given set of functions, and which have this property. Just how that could be done might be found out. Perhaps it is not possible. The notion of completeness of a set of functions is obviously easily transferable to the more general case. If it were possible to get for each complete sequence of functions a linearly equivalent set, in which the constants for the minimizing combination were independent of the number of functions (in order) of the sequence used, and a generalization of

[1] This method has been variously applied. *Cf.*, for instance, A. J. Pell, *Trans. Amer. Math. Soc.*, vol. 20 (1919), pp. 343–355; W. L. Hart, *Annals of Math.*, vol. 24 (1922), pp. 23–38. Levy, *loc. cit.*, pp. 117–129, 292–313.

the Riesz-Fischer theorem were obtained, then perhaps this matter of reducing the functional equation to the corresponding case in infinitely many variables might be a possibility. In any case, the minimal property of orthogonal functions is no doubt worth while.

A question which is of another kind is whether there is any relation between functional transformations and bilinear operations. In n-variables a system of algebraic equations and a bilinear form go together, similarly in infinitely many variables and in integral equations. Is there any relationship in this more general basis, and if so what?

We might continue further in this subject of functional transformations, but the above indications are no doubt sufficient to indicate that there are many unanswered questions, and many things to be done. To any one interested in other questions relating to this field, I am sure that there will be found much that is suggestive in the book of Levy on *Analyse Fonctionelle*, already referred to. This book, taken together with existing theories for instance of partial differential equations in n-space, is an admirable instance of the first part of Moore's tenet of faith for general analysis. Logically then there should be opportunity as stated in the second part of this statement for a general theory, and no doubt the near future will produce much of this. Among other things, perhaps there will be shown the possibility of generalizing the mean value integration of Gateaux as expounded by Levy. Perhaps the above suggestion of introducing an n-dimensional field into the reasoning *via* a sequence of functions which is complete for the class of functions, and the minimal process above referred to, may be a line of attack on this problem.

In closing, I wish to express to you my appreciation of the opportunity of presenting these ideas to you, and the hope that perhaps they may contain something which will be of value for further research in general analysis.

SOLUTIONS OF A PROBABILITY DIFFERENCE EQUATION.

By OTTO DUNKEL, Washington University.

1. Introduction. The writer has recently given solutions of two types of probability problems in coin tossing which were obtained by counting the number of favorable cases.[1] The first type will be considered here; the three cases under this type may be stated as follows:

I. *The probability, $p_1(n)$, of a sequence of a tail followed by r or more consecutive heads turning up in n tosses of a coin.*

II. *The probability, $p_2(n)$, of tossing at first r or more consecutive heads and in the subsequent tosses no set of a tail followed by as many as r consecutive heads.*

III. *The probability, $p_3(n)$, of r or more consecutive heads turning up in n tosses of a coin.*

[1] *Washington University Studies, Scientific Series*, vol. XII, Jan. 1925, no. 2, pp. 119–136. See also in the MONTHLY the solution of **3046** [*1924*, 403].

It is clear that

$$p_3(n) = p_1(n) + p_2(n), \tag{1}$$

and it was shown that

$$p_1(n) = 1 - 2^r p_2(n + r). \tag{2}$$

It suffices then to determine $p_2(n)$ in order to have a determination of $p_1(n)$ and $p_3(n)$. It was also shown that $p_2(n)$ satisfies the difference equation

$$p_2(n + 1) - p_2(n) = -\frac{p_2(n - r)}{2^{r+1}}, \tag{3'}$$

$$p_2(n) = 0, \qquad 0 \leqq n < r, \qquad p_2(r) = 1/2^r.$$

The object of the present paper is to discuss the solutions of a simple transformation of the equation (3'), and it will be foreseen from the simplicity of that equation that the discussion will be elementary. But it is believed that it is of sufficient interest to develop since it affords a simple illustration of important general theorems on difference equations, of a method of determining the real roots of algebraic equations similar to that of Bernoulli, and of a method of obtaining bounds for the absolute values of the imaginary roots. It will also be shown that the probability, $p_2(n)$, is related in a simple way to a generalization of the Fibonacci sequence of numbers. A determination of equation (3') will be given by a method different from that previously used since it shows that $p_2(n)$ satisfies also an equation of lower order but of non-homogeneous form. There will also be given a different determination of $p_2(n)$ which leads to additional information. The well-known fact that the ratio $p_2(n + 1)/p_2(n)$ approaches the largest real root of the characteristic algebraic equation will be shown in two ways.

If the finite sum which defines $p_2(n)$ is made an infinite series, this infinite series will be shown to be a solution; and it will be further shown that the ratio of the two series for the values $n + 1$ and n is precisely a root of the algebraic equation.[1] Most of the theorems and proofs given here may be generalized.

2. Derivation of the Difference Equation. Let $P_2(n)$ be the number of favorable cases for probability II when there are n tosses of a coin, and consider the number of favorable cases when an additional toss, the $(n + 1)$th, is made. If the $(n + 1)$th toss is a tail, then the number of favorable cases is $P_2(n)$. Suppose now that it is a head. There is just one case in which all the n tosses are heads. If they are not all heads, there will be favorable cases when the first $n - k - 1$ of the n tosses are favorable while the last $k + 1$ tosses consist of a tail followed by k heads, $0 \leqq k < r - 1$. For each value of k there are $P_2(n - k - 1)$ favorable cases. Summing up these results we have the theorem.

[1] This result bears a certain relation to the theorem in Problem **3071** [*1924*, 206], E. L. Post, of this MONTHLY. It seems probable that this theorem may be derived from the results given here by a limiting process.

THEOREM. *The number of favorable cases in* II, $P_2(n)$, *satisfies the non-homogeneous equation of the rth order*

$$P_2(n+1) = P_2(n) + P_2(n-1) + \cdots + P_2(n-r+1) + 1, \qquad (4)$$
$$P_2(n) = 0, \quad 0 \leqq n < r; \qquad P_2(r) = 1.$$

COROLLARIES. Replace in (4) n by $n-1$ and subtract this result from (4). There results the

THEOREM. $P_2(n)$ *must satisfy the homogeneous equation*

$$P_2(n+1) = 2P_2(n) - P_2(n-r), \qquad (3)$$

with the same initial conditions.

In order to pass to the corresponding equations for $p_2(n)$ we have merely to divide both sides of (3) and (4) by 2^{n+1}. In the case of (3) we find the equation and conditions given in (3′), and from (4) we have

$$p_2(n+1) = \frac{p_2(n)}{2} + \frac{p_2(n-1)}{2^2} + \cdots + \frac{p_2(n-r+1)}{2^r} + \frac{1}{2^{n+1}}. \qquad (5)$$

3. The Homogeneous Linear Difference Equation. We shall here state a few facts, which will be referred to later, regarding the difference equation

$$P(n+r) = A_{r-1}(n)P(n+r-1) + A_{r-2}(n)P(n+r-2) + \cdots + A_0(n)P(n), \qquad (6)$$

where r is a positive integer, $A_i(n)$ and $P(n)$ are functions of the integer n, and $A_0(n) \neq 0$. All of the statements are easily proved.

If we have r solutions of (6), $P^{(1)}(n)$, $P^{(2)}(n)$, $\cdots$, $P^{(r)}(n)$, the determinant

$$\Delta(n) = \begin{vmatrix} P^{(1)}(n) & P^{(2)}(n) & \cdots & P^{(r)}(n) \\ P^{(1)}(n+1) & P^{(2)}(n+1) & \cdots & P^{(r)}(n+1) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ P^{(1)}(n+r-1) & P^{(2)}(n+r-1) & \cdots & P^{(r)}(n+r-1) \end{vmatrix}, \qquad (7)$$

will be called the *determinant* of these solutions. This determinant satisfies the relation

$$\Delta(n+m+1) = (-1)^{(m+1)(r-1)} A_0(n+m)A_0(n+m-1) \cdots A_0(n)\Delta(n), \qquad (8)$$

and, therefore, if it is not zero for one value of n, it never vanishes. In this case the r solutions are linearly independent; and if $P(n)$ is any solution, then

$$P(n) = C_1P^{(1)}(n) + C_2P^{(2)}(n) + \cdots + C_rP^{(r)}(n), \qquad (9)$$

where the C's are constants.

If the A's in (6) are constants, solutions may be obtained by solving the *characteristic algebraic equation*

$$x^r = A_{r-1}x^{r-1} + A_{r-2}x^{r-2} + \cdots + A_1x + A_0, \qquad A_0 \neq 0, \tag{10}$$

and then setting $P(n) = a^n$, where a is a root. If the r roots of (10) are distinct, the determinant (7) is not zero for the r solutions thus obtained and they are linearly independent solutions. If a is a p-fold root, then there are corresponding to it the p solutions

$$a^n, \quad na^n, \quad n^2a^n, \quad \cdots, \quad n^{p-1}a^n.$$

The r solutions obtained in this way are again linearly independent. If the A's are real, then to each pair of conjugate imaginary roots there corresponds a pair of real solutions.

If $P(n)$ is a solution such that

$$P(n) = 0, \qquad 1 \leqq n \leqq r-1, \qquad P(r) = 1, \tag{11}$$

then $P(n+r-1)$, $P(n+r-2)$, $\cdots$, $P(n)$ is a system of linearly independent solutions for which

$$\Delta(n) = (-1)^{(n-1)(r-1)}A_0^{n-1}. \tag{11'}$$

4. The Probability Algebraic Equation. The characteristic algebraic equation for (3') is

$$x^{r+1} - x^r + \frac{1}{2^{r+1}} = 0. \tag{12'}$$

If the roots of this equation are multiplied by 2, there results the equation

$$f(x) \equiv x^{r+1} - 2x^r + 1 = 0, \tag{12}$$

which corresponds to the difference equation (3) for $P_2(n)$. For simplicity we consider this second equation. If $r = 1$, this equation has the double root 1, and we shall exclude this case in what follows.

THEOREM. *There are only two positive roots, unity and a root between one and two. These are the only real roots if r is odd; if r is even, there is only one more real root and it lies between -1 and 0.*

There are no multiple roots and no two roots have equal absolute values unless they constitute a pair of conjugate imaginaries. The absolute values of the imaginary roots are less than unity.

PROOF. The first statement follows from the facts that $f(0) = 1$, $f(1) = 0$, $f(2) = 1$, and that the derivative

$$f'(x) = x^{r-1}[(r+1)x - 2r] \tag{13}$$

vanishes only at $x = 0$ and at $x = 2r/(r+1)$. Since $f'(1) = 1 - r < 0$ and

$f'(2) = 2^r$, it is obvious that first $f'(x)$ and then $f(x)$ vanishes between 1 and 2. The facts regarding the negative root and the non-existence of multiple roots are easily verified. It has been shown above that $f[2r/(r+1)]$ is actually negative, and we shall set down here a result which will be used later. By a simple transformation of this inequality it becomes

$$\frac{r+1}{2^{r+1}}\left(\frac{r+1}{r}\right)^r < 1, \qquad r \geqq 2. \tag{14}$$

In order to examine the imaginary roots we set $x = \rho(\cos\theta + i\sin\theta)$, where $0 < |\theta| < \pi$. This gives the pair of equations

$$\rho^{r+1}\cos(r+1)\theta - 2\rho^r\cos r\theta = -1, \qquad \rho^{r+1}\sin(r+1)\theta - 2\rho^r\sin r\theta = 0.$$

Squaring both sides and adding the results, we find

$$\rho^{2(r+1)} - 4\rho^{2r+1}\cos\theta + 4\rho^{2r} = 1.$$

For a given value of ρ, $\cos\theta$ can have only one value, and hence θ can have only the two values corresponding to a single conjugate pair of roots with the absolute value ρ. This shows also that no imaginary root can have an absolute value equal to that of a real root.

In order to prove the last statement consider the equation whose roots are the reciprocals of the roots of (12),

$$y^{r+1} - 2y + 1 = 0. \tag{15}$$

This equation has the root 1 and a second positive root, say b, which is less than 1. Then, since $y^{r+1} - b^{r+1} = 2(y-b)$, we have

$$y^r + y^{r-1}b + \cdots + b^{r-1}y + b^r = 2, \qquad y \neq b.$$

Suppose that y is an imaginary root such that $|y| = \rho \leqq 1$, then

$$2 = |y^r + y^{r-1}b + \cdots + b^{r-1}y + b^r| < \rho^r + b\rho^{r-1} + \cdots + b^{r-1}\rho + b^r$$
$$< 1 + b + \cdots + b^{r-1} + b^r = 2.$$

Since y is imaginary, the equality sign is excluded above. The last equality on the right follows since b is a root of the equation obtained from (15) by removing the root unity. From the contradiction above it follows that we must have $\rho > 1$; and this means that the modulus of an imaginary root of (12) must be less than unity.

REMARKS. It will be shown later how some of these results may be obtained in a different way.

It may be shown that the absolute values of the imaginary roots of (12) are greater than $1/\sqrt[r]{3} > 1/2$.

If r is even, the absolute values of the imaginary roots lie in an interval whose bounds are unity and the absolute value of the negative root. The length of this interval is less than .23 and it decreases as r increases.

An approximation, in excess, to the larger positive root of (12) is given by

$$2\left[1 - \frac{1}{2^{r+1} - r}\right], \tag{16}$$

which is obtained by applying Newton's method to (15) rather than to (12). It will now be shown how to approximate this root in a different manner.

5. Approximations to the Larger Positive Root.

THEOREM. *The ratio*

$$\frac{P_2(n+1)}{P_2(n)} \tag{17}$$

decreases as n increases from $2r-1$ and approaches as a limit the larger positive root a_1 of equation (12).

PROOF. For convenience set the ratio (17) equal to $R(n)$, then we may write (3) in the form

$$R(n) = 2 - \frac{1}{R(n-1)R(n-2)\cdots R(n-r)}. \tag{18}$$

From the conditions imposed upon $P_2(n)$ we have

$$R(n) = 2, \qquad r \leqq n \leqq 2r-1; \qquad R(2r) = 2\left[1 - \frac{1}{2^{r+1}}\right].^{1}$$

If in (18) the denominator decreases, the whole expression, *i.e.*, $R(n)$, decreases. Now from the values of $R(n)$ above this happens when n in (18) increases from $2r$ to $2r+1$, and hence $R(n)$ will continue to decrease as n increases from this point on. It now follows easily that $R(n)$ approaches a limit $\bar{a}_1$ which is a root of (12).

It remains to show that $\bar{a}_1$ is not 1. If each factor in the denominator of (18) is greater than a_1, the larger positive root of (12), then the whole expression on the right, *i.e.*, $R(n)$, is such that $R(n) > 2 - 1/a_1^r = a_1$. Now when n in (18) is equal to $2r$, this is true, for then each factor is 2. Hence $R(2r) > a_1$, and it then follows that every $R(n)$ is greater than a_1. Therefore $\bar{a}_1 = a_1$.

REMARKS. From this it follows at once that

$$\operatorname*{Limit}_{n=\infty} \frac{p_2(n+1)}{p_2(n)} = \frac{a_1}{2} = c_1, \tag{19}$$

where c_1 is the larger positive root of (12′). This would furnish an approxima-

[1] This expression is also given as an approximation to a_1 when Newton's method is applied to (12).

tion formula for $p_2(n)$ for large values of n. Such a formula will be furnished in another way later. The above results will be shown in a second way which applies more generally but does not prove so much.

6. Two Expressions for $P_2(n)$ and $p_2(n)$. The evaluation of probability II is furnished by the following theorem.

THEOREM. *If* $[(n+1)/(r+1)]=i$ *means the greatest integer in* $(n+1)/(r+1)$,

$$P_2(n) = \sum_{k=1}^{k=i} (-1)^{k+1}{}_{n-kr}C_{k-1}2^{n-k(r+1)+1},$$

$$p_2(n) = \frac{1}{2^r}\sum_{k=1}^{k=i}(-1)^{k+1}{}_{n-kr}C_{k-1}\left(\frac{1}{2^{r+1}}\right)^{k-1}, \qquad {}_mC_p = \frac{m!}{p!(m-p)!}. \tag{20}$$

If $a_1, a_2, \cdots, a_{r+1}$ *are the roots of* (12), *then*

$$P_2(n) = \sum_{k=1}^{k=r+1} \frac{a_k{}^n}{f'(a_k)}. \tag{21}$$

PROOF. Since $a_1{}^n, a_2{}^n, \cdots, a_{r+1}{}^n$ are linearly independent solutions of (3), we must have

$$P_2(n) = \sum_{k=1}^{k=r+1} C_k a_k{}^n,$$

where the constants C_k are determined uniquely by the equations

$$1 = \sum_{k=1}^{k=r+1} C_k a_k{}^r, \qquad 0 = \sum_{k=1}^{k=r+1} C_k a_k{}^n, \qquad 0 \leqq n \leqq r-1.$$

It is obvious that no C_k can be zero. One of the roots of (12) is 1 and the corresponding C_k is $1/(1-r)$. For subtract from each term of the first equation the sum of the corresponding terms of the last r equations and note that $a_k{}^r = a_k{}^{r-1} + a_k{}^{r-2} + \cdots + a_k + 1$, if $a_k \neq 1$. This is what we should expect, for $1/(1-r)$ is a solution of the non-homogeneous equation (4), and since $P_2(n)$ is also a solution, it must necessarily contain this term plus a solution of the homogeneous equation obtained by dropping the constant term 1 in (4).

The coefficients C_k are the quotients of certain determinants formed from the roots a_k which are easily written. But to avoid the labor of the reduction of these determinants and at the same time to obtain other results, we shall employ a method which is often used for this purpose. We set

$$\frac{\varphi(x)}{f(x)} = \sum_{n=1}^{n=\infty} \frac{P_2(n)}{x^n},$$

where $\varphi(x)$ is a polynomial in x of degree not higher than r, and we determine it so that $P_2(n) = 0$, $1 \leqq n \leqq r-1$, and $P_2(r) = 1$, $P_2(r+1) = 2$. Multiplying

both sides by $f(x)$, we have

$$\varphi(x) = \Sigma[P_2(n+r+1) - 2P_2(n+r) + P_2(n)]x^{-n}$$
$$= A_r x^r + A_{r-1}x^{r-1} + \cdots + A_1 x + A_0.$$

This gives

$$A_0 = P_2(r+1) - 2P_2(r) = 0, \qquad A_1 = P_2(r) = 1,$$
$$A_2 = A_3 = \cdots = A_r = 0,$$
$$P_2(n+r+1) - 2P_2(n+r) + P_2(n) = 0, \qquad n \geqq 1.$$

Thus $\varphi(x) = x$, and, if we suppose that $x \geqq 3$, we may write

$$\frac{x}{f(x)} = \frac{1}{x^r\left[1 - \left(\frac{2}{x} - \frac{1}{x^{r+1}}\right)\right]} = \sum_{n=0}^{n=\infty} \frac{1}{x^r}\left(\frac{2}{x} - \frac{1}{x^{r+1}}\right)^n.$$

We may develop each term of the series by the binomial theorem and rearrange the terms of the result. This gives after a slight change in the notation

$$\sum_{n=r}^{n=\infty} \frac{1}{x^n} \sum_{k=1}^{k=i} (-1)^{k+1}{}_{n-kr}C_{k-1}2^{n-k(r+1)+1}, \qquad i = \left[\frac{n+1}{r+1}\right];$$

and, since $P_2(n)$ is the coefficient of x^{-n}, we have the first part of the result in (20). The expression for $p_2(n)$ is now obtained by dividing $P_2(n)$ by 2^n.

Returning to the rational function of x above, we may write it by aid of the theory of partial fractions in the form

$$\frac{x}{f(x)} = \sum_{k=1}^{k=r+1} \frac{a_k}{f'(a_k)} \frac{1}{x - a_k} = \sum_{k=1}^{k=r+1} \frac{a_k}{f'(a_k)x} \frac{1}{\left(1 - \frac{a_k}{x}\right)}.$$

Since $|a_k/x| < 2/3$, we may develop each term of the finite sum in an infinite series and collect the terms thus:

$$\frac{x}{f(x)} = \sum_{n=1}^{n=\infty} \frac{1}{x^n} \sum_{k=1}^{k=r+1} \frac{a_k{}^n}{f'(a_k)}.$$

From this the expression (21) now follows.

REMARKS. The expressions for $P_2(n)$ and $p_2(n)$ in (20) define these functions only for positive values of n. However, the difference equations define them for negative values also. Expressions corresponding to (20) when n is negative may be obtained without difficulty by similar developments. On the other hand, the solution of the difference equation given by (21) is valid for both positive and negative values of n. The expression for $p_2(n)$ corresponding to (21) is easily written.

7. The Rapidity of Convergence of $P_2(n+1)/P_2(n)$. The proof of formula (21) applies to the general linear homogeneous difference equation with constant coefficients provided that the roots of the algebraic equation are distinct. In case there are equal roots there is no difficulty in suitably modifying the discussion. Thus a second and more general proof that the limit of the ratio $P_2(n+1)/P_2(n)$ is a_1 may be obtained from (21). This shows also that for sufficiently large values of n we may write

$$P_2(n) = \frac{{a_1}^n}{f'(a_1)} \text{ approximately,} \tag{22}$$

if the absolute value of a_1 is greater than that of any other root.

We shall now prove the following theorem.

THEOREM. *The difference*

$$\frac{P_2(n+2)}{P_2(n+1)} - \frac{P_2(n+1)}{P_2(n)} = \frac{P_2(n+2)P_2(n) - P_2^2(n+1)}{P_2(n+1)P_2(n)} \tag{23}$$

is negative after a sufficiently large value of n and it approaches zero like $1/{a_1}^n$.

PROOF. The numerator of (23) is

$$\begin{vmatrix} P_2(n+2) & P_2(n+1) \\ P_2(n+1) & P_2(n) \end{vmatrix} = \sum\sum C_iC_j(a_ia_j)^n(a_i - a_j)^2, \qquad j > i = 1, 2, \cdots, r,$$

while the denominator is

$$P_2(n+1)P_2(n) = \sum_{i=1}^{r+1} C_i^2 a_i^{2n+1} + \sum\sum C_iC_j(a_ia_j)^n(a_i + a_j), \qquad j > i,$$

where the values of the C's are given in (21). Hence for large values of n (23) is approximately

$$\frac{C_2{a_2}^n(a_1 - a_2)^2}{C_1{a_1}^{n+1}} = \frac{f'(a_1)(a_1 - 1)^2}{f'(1){a_1}^{n+1}} = \frac{[(r+1)a_1 - 2r](a_1 - 1)^2{a_1}^{r-2}}{(1-r){a_1}^n}, \tag{24}$$

since $a_1 > 1$ $(r \geqq 2)$ and the absolute values of the remaining $r-1$ roots are less than the root $a_2 = 1$. Since here a_1 and 1 are consecutive simple roots, $f'(a_1)/f'(1)$ is negative; hence for sufficiently large values of n the ratio $P_2(n+1)/P_2(n)$ decreases. The previous proof determined this value of n.

REMARKS. In the case of the general equation, if $P_2(n+1)/P_2(n)$ approaches a limit a_1, and no two roots of the algebraic equation have equal absolute values unless they form a conjugate pair of imaginaries, it may be shown by the use of the formula for (23) that a_1 is a root with a greater absolute value than any other root. If $P_2(n+1)/P_2(n)$ ultimately always decreases, it may be shown by (23) that the root whose absolute value is next in magnitude to that of a_1 is the real root a_2. For simplicity of statement we have assumed that a_1 and a_2 are both positive and that the coefficients in the difference equation are real. A

second proof of the first part of this theorem which does not require so much is given later.

8. The Solution of Numerical Equations. If it is known that a given numerical equation with real coefficients has a real root a_1 whose absolute value is greater than that of any other root, this root may be approximated by the ratio $P(n+1)/P(n)$, where $P(n)$ is a solution of the corresponding difference equation similar to $P_2(n)$. The initial values of $P(n)$ may be chosen arbitrarily, but in this case, if the algebraic equation is reducible, the approximation may lead to the real root of the reduced equation whose absolute value is greater than that of its other roots. The method also applies when a_1 is a multiple root. Also if a_1 and $-a_1$ are two real roots of greater absolute value than those of the remaining roots, a slight change in the equation permits the method to be applied to find a_1. If the root a_2 whose absolute value is next in magnitude to that of a_1 is real, the ratio a_2/a_1 may be approximated by the ratio of two of the expressions (23) for the values $n+1$ and n. Then a_2 may be found after a_1 has been determined.

Theoretically, any real root may be approximated in this way after making the transformation of the equation by $y = 1/(x-c)$, where c is chosen near enough to the desired root to make the corresponding root of the y-equation have the greatest absolute value. However, this would not in general give a good practical process. If the equation has a real root of absolute value smaller than that of any other root, then $y = 1/x$ transforms the equation and the root so that the method may be applied to the resulting equation.

The equation (12) becomes after the transformation $y = 1/x$

$$y^{r+1} - 2y + 1 = 0, \tag{25}$$

and the corresponding ratio is

$$\frac{P_2(-n-1)}{P_2(-n)} = \frac{Q(n+1)}{Q(n)}, \tag{26}$$

where $Q(n)$ satisfies the equation

$$Q(n+r+1) - 2Q(n+1) + Q(n) = 0. \tag{27}$$

If r is odd, the ratio (26) cannot have a limit as we see from the nature of the roots. If r is even, (25) has a third real root of absolute value greater than 1. From the facts stated above, page 359, regarding the roots it follows that (26) must approach this root as a limit. But this process is useless as a practical method since the convergence is extremely slow due to the very slight difference between the absolute value of this root and that of an imaginary root.

9. Relations between Powers of a Root.

THEOREM. *If x is any root of* (12),

$$x^n = P_2(n)x^r - \sum_{k=0}^{k=r-1} P_2(n-k-1)x^k. \tag{28}$$

PROOF. Writing the equation (12) in the form $x^{r+1} = 2x^r - 1$, we multiply both sides by $x, x^2, x^3, \cdots$, and after each multiplication we reduce the right side to the degree r by means of the original equation above. The result may be written

$$x^n = \sum_{k=0}^{k=r} F_k(n)x^k, \tag{29}$$

$$F_k(n) = 0, \qquad n \neq k, \qquad F_k(k) = 1, \qquad 0 \leqq n \leqq r.$$

Multiply both sides of (29) by x and reduce the right side to the degree r as before; we obtain

$$x^{n+1} = [2F_r(n) + F_{r-1}(n)]x^r + \sum_{k=0}^{r-2} F_k(n)x^{k+1} - F_r(n).$$

Comparing this with (29), we find

$$F_r(n+1) = 2F_r(n) + F_{r-1}(n), \qquad F_{k+1}(n+1) = F_k(n),$$
$$F_0(n+1) = -F_r(n), \qquad 0 \leqq k \leqq r-2.$$

From these equations we obtain in turn

$$F_k(n) = -F_r(n-k-1), \qquad F_r(n+1) = 2F_r(n) - F_r(n-r), \qquad 0 \leqq k \leqq r-1.$$

The last equation shows that $F_r(n)$ is a solution of (3) and from the initial conditions in (29) we see that $F_r(n) = P_2(n)$. Inserting the values of $F_k(n)$ in (29), we obtain (28).

REMARKS. Setting $x = 1$ in (28), it follows that if $P_2(n)$ is a solution of (3) with the given initial conditions then it must also satisfy (4).

The equation (28) furnishes a proof of the

THEOREM. *Given that the roots of* (12) *are distinct, that $P_2(n)$ is a solution of* (3) *with the given initial conditions, that $P_2(n+1)/P_2(n)$ approaches a limit a_1 as n becomes infinite, then a_1 is a root of* (12) *whose absolute value is greater than that of any other root.*

This theorem and its proof may be generalized.

PROOF. It follows that a_1 is a root of (12) by the same reasoning indicated before. Divide both sides of (28) by $P_2(n-r)$ and take the limit of each side for $n = \infty$; we have as the result

$$\operatorname*{Limit}_{n=\infty} \frac{x^n}{P_2(n-r)} = a_1{}^r x^r - \sum_{k=0}^{k=r-1} a_1{}^{r-k-1}x^k = a_1{}^r x^r - \frac{a_1{}^r - x^r}{a_1 - x}, \tag{30}$$

where in the last form to the right $x \neq a_1$. If $x \neq a_1$, the right side of (30) is zero; for it is equal to

$$\frac{x^r f(a_1) - a_1{}^r f(x)}{a_1 - x}.$$

If $x = a_1$, the right side reduces to $a_1{}^r f'(a_1)$, and then (30) gives after a slight change

$$\operatorname{Limit}_{n=\infty} \frac{a_1{}^n}{P_2(n)} = f'(a_1).$$

If $x \neq a_1$, we have now

$$\operatorname{Limit}_{n=\infty} \left(\frac{x}{a_1}\right)^n = \frac{1}{a_1{}^r f'(a_1)} \operatorname{Limit}_{n=\infty} \frac{x^n}{P_2(n-r)} = 0.$$

This shows that $|x| < |a_1|$.

If both sides of (28) are multiplied by $1/f'(x)$ and the sum of each side is taken for $x = a_1, a_2, \cdots, a_{r+1}$, we obtain again the result (21) making use of the known relations

$$\sum_{i=1}^{i=r+1} \frac{a_i{}^r}{f'(a_i)} = 1, \qquad \sum_{i=1}^{i=r+1} \frac{a_i{}^m}{f'(a_i)} = 0, \qquad 0 \leqq m < r.$$

From the equation (28) relations between the functions $P_2(n)$ for different values of n may be obtained. One procedure is to multiply both sides by x^m, then reduce the right side to the degree r and compare the result with the expression for x^{n+m} given directly by (28).

10. The Fibonacci Sequence. Since $P_2(n)$ is a solution of the non-homogeneous equation (4), it is at once evident that

$$P_2(n+1) - P_2(n) = F(n) \tag{30A}$$

must be a solution of the homogeneous equation

$$F(n+r) = F(n+r-1) + F(n+r-2) + \cdots + F(n), \tag{31}$$
$$F(n) = 0, \qquad 0 \leqq n \leqq r-2, \qquad F(r-1) = 1.$$

When $r = 2$, this gives the series of numbers usually called the Fibonacci series. Other relations are given in the following theorem.

THEOREM. *If $F(n)$ is defined as in* (30),

$$P_2(n) = \frac{1}{r-1}\left[F(n+r-1) - \sum_{k=3}^{k=r} (k-2)F(n+r-k) - 1\right],$$

$$= \frac{1}{r-1}\left[F(n+r) - \sum_{k=2}^{k=r} (k-1)F(n+r-k) - 1\right], \tag{32}$$

$$= \sum_{n=r-1}^{n-1} F(n). \tag{33}$$

PROOF. Write (4) in the equivalent form

$$P_2(n+r) - P_2(n+r-1) = \sum_{k=3}^{k=r} (k-2)[P_2(n+r-k+1) - P_2(n+r-k)] + (r-1)P_2(n) + 1.$$

Then from (30A) we have at once the first equation in (32). The second form follows by use of (31).

If we multiply both sides of

$$P_2(n-k) = P_2(n-k-1) + \cdots + P_2(n-k-r) + 1$$

by $F(r+k-1)$, and take the sum of the corresponding sides from $k=0$ to $k=n-r$, there results after certain cancellations by use of (31) the equation given in (33).

REMARKS. From (11′) we may obtain a theorem regarding the determinant of the $F(n)$'s which is well known in the case of $r=2$. Other relations involving this function may be obtained by the method indicated at the end of the previous section.

11. An Infinite Series as a Solution. It will be more convenient now to consider the equation (3′). It will be readily seen that the second expression of (20) is formally a solution of this equation when the upper limit i in the finite sum is made infinite. It remains to show that

$$p(n) = \frac{1}{2^r}\left[1 + \sum_{k=2}^{\infty} (-1)^{k+1} \frac{(n-kr)(n-kr-1)\cdots(n-kr-k+2)}{(k-1)!} \left(\frac{1}{2^{r+1}}\right)^{k-1}\right], \quad (34)$$

is convergent. Here we shall have a solution for both positive and negative values of n. To show the convergence of (34) it suffices to examine the infinite series

$$q(n) = 1 + \sum_{k=2}^{\infty} \frac{(n+kr)(n+kr+1)\cdots(n+kr+k-2)}{(k-1)!} \left(\frac{1}{2^{r+1}}\right)^{k-1}, \quad (34')$$

where n is positive. The ratio of two consecutive terms of this series for $k \geqq r+2$ may be written after certain cancellations

$$\left(1+\frac{k-1}{n+kr}\right)\left(1+\frac{k-1}{n+kr+1}\right)\cdots\left(1+\frac{k-1}{n+kr+r-1}\right)\left(r+1+\frac{n+r-1}{k}\right)\frac{1}{2^{r+1}}.$$

This ratio is less than

$$\left(1+\frac{1}{r}\right)^r\left(r+1+\frac{n+r-1}{k}\right)\frac{1}{2^{r+1}}.$$

Hence the series (34′) converges if

$$\frac{r+1}{2^{r+1}}\left(1+\frac{1}{r}\right)^r < 1,$$

and this has already been shown to be true if $r \geqq 2$ in (14). Thus the series (34) converges for all values of n.

THEOREM. *If c_1 is the larger positive root of (12'), then*

$$\frac{p(n+1)}{p(n)} = c_1, \qquad p(n) = Ac_1^n, \tag{35}$$

where A is a constant and n is an integer.

PROOF. The function $q(n) = 2^r p(-n)$ defined in (34') satisfies the equation

$$q(n+1) = q(n) + \frac{q(n+r+1)}{2^{r+1}}, \tag{36}$$

for which the algebraic equation is

$$y = 1 + \frac{y^{r+1}}{2^{r+1}}. \tag{36'}$$

The roots of this equation are the reciprocals of $c_1, c_2, \cdots, c_{r+1}$, the roots of (12').

Consider the two terms in the ratio

$$\frac{q(n+1)}{q(n)}, \qquad n \geqq 0, \tag{37}$$

one in the numerator and the other in the denominator, which correspond to the same value of k. The ratio of these two terms is

$$\frac{(n+1+kr)(n+2+kr)\cdots(n+k-1+kr)}{(n+kr)(n+1+kr)\cdots(n+k-2+kr)} = 1 + \frac{k-1}{n+kr} < 1 + \frac{1}{r}.$$

Hence the ratio (37) lies between 1 and $1 + 1/r$. If we set this ratio equal to $R(n)$, we may write (36) in the form

$$R(n) = 1 + \frac{R(n+r)R(n+r-1)\cdots R(n)}{2^{r+1}},$$

and this enables us to determine a series of upper bounds l_k for $R(n)$ and also a series of lower bounds m_k, where $m_1 = 1$ and $l_1 = 1 + 1/r$. Thus we have at once

$$m_2 = 1 + \frac{1}{2^{r+1}} < R(n) < 1 + \frac{1}{2^{r+1}}\left(\frac{r+1}{r}\right)^{r+1} = l_2.$$

Obviously $m_2 > m_1$, and since (14) may be written in the form

$$1 + \frac{1}{2^{r+1}}\left(\frac{r+1}{r}\right)^{r+1} < 1 + \frac{1}{r},$$

we see that $l_2 < l_1$. Each of the bounds m_k and l_k satisfies the relation

$$x_{k+1} = 1 + \frac{x_k^{r+1}}{2^{r+1}};$$

and it follows at once that as k increases l_k decreases and m_k increases. Hence each approaches a limit which must be a root of (36′) which lies between 1 and 2. Since there is only one such root, $1/c_1$, both l_k and m_k approach this root as a common limit. It now follows that the ratio (37) is equal to $1/c_1$. From this follows that $p(-n) = 2^{-r}q(n) = Ac_1^{-n}$, $n \geqq 0$. Since $p(n)$ is linearly dependent upon the $(r+1)$ solutions c_1^n, c_2^n, $\cdots$, c_{r+1}^n, we have for both positive and negative values of n

$$p(n) = Ac_1^n + A_2c_2^n + \cdots + A_{r+1}c_{r+1}^n.$$

But for negative values of n, $A_2 = A_3 = \cdots = A_{r+1} = 0$, and the proof of (35) is complete.

REMARKS. A result similar to the above is given by the series [1]

$$f(x) = \sum_{n=0}^{\infty} \frac{(x+nh)^n}{n!},$$

which satisfies the equation

$$f'(x) = f(x+h), \tag{40}$$

for which the corresponding equation is

$$e^{mh} = m. \tag{40'}$$

Here it may be shown that $f(x) = Ae^{mx}$, where A is a constant. A root of the equation (40′) is given by $f(x+h)/f(x)$. In this case the quotient may be evaluated in the form of a single series not containing x. It may also be possible to evaluate the quotient defining c_1 in (35) as a single series.

It will be seen that the methods used above may be applied to determine a solution in the form of an infinite series similar to (34) for the equation

$$p(x+d) - p(x) = dp(x+r+d);$$

and then by a limit process we may pass to the solution of (40) and (40′). This limit process, however, has been examined by the writer only in a formal way, but it seems likely that the formal steps can be justified so as to furnish the theorems above regarding (40).

12. Numerical Illustrations. A small table of values of $p_1(n)$, $p_2(n)$ and $p_3(n)$ for $r = 5$ was given in a previous paper.[2] These values were computed by first calculating $p_1(n)$ by a formula similar to that for $p_2(n)$ (20). These results could have been computed more easily and quickly by using the recurrent relation (4) to calculate a table of values of $P_2(n)$; from the values of $P_2(n)$ those for $p_2(n)$ may then be obtained by division by 2^n. Below is given a table of values of $P_2(n)$ for $r = 5$ calculated in this way. After writing down the first five values

[1] Post, Problem **3071** [*1924*, 206].

[2] Dunkel, *Washington University Studies*, l.c., p. 126.

1, 2, 4, 8, 16, these are added together with 1 to give the next entry 32. A slip of paper may be used to cover all of the entries in a column except the last five, and a figure 1 should be marked on the edge of the slip in line with the unit column. After the addition of this 1 and the last five entries exposed the slip is advanced one space to leave exposed the next set of five, and so on. The preparation of the table below including the time for the verification of each entry takes about twenty minutes.

$r = 5$

n	$P_2(n)$	n	$P_2(n)$
5	1	23	210687
6	2	24	414200
7	4	25	814296
8	8	26	1600864
9	16	27	3147216
10	32	28	6187264
11	63	29	12163841
12	124	30	23913482
13	244	31	47012668
14	480	32	92424472
15	944	33	181701728
16	1856	34	357216192
17	3649	35	702268543
18	7174	36	1380623604
19	14104	37	2714234540
20	27728	38	5336044608
21	54512	39	10490387488
22	107168	40	20623558784

From the last two entries we have

$$\frac{P_2(40)}{P_2(39)} = 1.9659482366682240\cdots$$

as an approximation to a_1. The correct figures may be determined by calculating $P_2(41)/P_2(40)$. This does not require an extension of the table, for

$$\frac{P_2(41)}{P_2(40)} = 2 - \frac{P_2(35)}{P_2(40)} = 1.9659482366571560\cdots$$

and therefore

$$a_1 = 1.96594823665\cdots.$$

The following values were also computed from the table by division and the use of (1) and (2):

$$p_2(35) = .020438704611, \qquad p_3(35) = .420214146347,$$
$$p_1(35) = .399775441736, \qquad p_2(40) = .018757017446.$$

From (22) we have approximately for large values of n, using the value of a_1 above,

$$p_2(n) = \left(\frac{a_1}{2}\right)^{n-4} \frac{1}{2^5(3a_1 - 5)} = .0348055734513(.98297411833)^{n-4}.$$

This formula gives by the aid of (1) and (2)

$$p_2(20) = .0264437, \qquad p_2(100) = .0066942,$$
$$p_2(50) = .0157974, \qquad p_1(100) = .803415,$$
$$p_3(100) = .810109.$$

All of these are perhaps correct in the figures given except $p_2(20)$ which may be incorrect in the last figure 7. The computation of $p_2(100)$ by the formula (20) would be quite tedious.

For $r = 2$, we have the exact formula

$$p_2(n) = \frac{1}{2^n}\left[\frac{5 + \sqrt{5}}{10}\left(\frac{1 + \sqrt{5}}{2}\right)^n + \frac{5 - \sqrt{5}}{10}\left(\frac{1 - \sqrt{5}}{2}\right)^n - 1\right].$$

For $r = 3$,

$$p_2(n) = C_1 c_1^n + C_2 c_2^n + \rho^n [C_3 \cos n\theta + C_4 \sin n\theta],$$

where

$$c_1 = .91964338, \quad c_2 = 1/2, \quad \rho = -.368676, \quad \theta = 55^\circ\ 18'\ 39.6'',$$
$$C_1 = .21780819, \quad C_2 = -1/2, \quad C_3 = .282164, \quad C_4 = -.359184.$$

AN ALGEBRA WITH SINGULAR ZERO.

By E. T. BELL, University of Washington.

1. To the many existing "algebras" which have been devised by modifying or abolishing certain of the restrictions imposed by the fundamental laws of common algebra, we propose to add another, distinguished by the singular properties of its zero. In common algebra, $b - b = 0$, and if this be taken as the definition of (additive) zero, then $c + 0 = 0 + c = c$. In the modification, if β, γ are any elements of the algebra, we do not have $\gamma + \beta - \beta = \gamma$, although we do have $\beta - \beta + \gamma = \gamma + \beta - \beta$. Our purpose in calling attention to this algebra is not to exhibit a barren freak, but to point out an extremely powerful method of dealing symbolically with certain classes of numbers and functions. Its use greatly simplifies, for example, the deduction of general relations (general in the sense that arbitrary functions are involved) between the Bernoulli numbers, or the Bernoulli polynomials of one or more variables, or the Bessel coefficients. Here merely the algebra itself is stated; the applications are developed in other papers. From the interpretation given it will be easy for any one who is interested to construct his own applications.

2. For brevity we shall refer to common algebra as A, and to the algebra which is to be constructed with singular zero as U. The elements of A are ordinary algebraic quantities $a, b, c, \cdots$, or *ordinaries;* the elements of U are *umbræ* $\alpha, \beta, \gamma, \cdots$ of ordinaries. The umbra α is a representative of the entire class of ordinaries α_i $(i = 0, 1, 2, \cdots)$, and similarly for $\beta, \gamma, \cdots$. We say that α is an umbra of the class α_i $(i = 0, 1, 2, \cdots)$. It is assumed unless otherwise stated that umbræ denoted by different letters are distinct; thus α_i, β_i $(i = 0, 1, 2, \cdots)$ are different classes of ordinaries, so that at least one of the classes contains at least one element not in the other.

3. It will be of assistance here to recall a few points of the "representative" or umbral calculus of Blissard.[1] The nth power, α^n, of the umbra α is purely symbolic, or umbral, and represents the ordinary, α_n. Operations upon umbral powers are to be performed as if the powers were ordinary until the last step,

[1] *Quarterly Journal of Mathematics*, vols. 4, 5; 3 papers; also Lucas, *Théorie des Nombres*, chap. XIII.

when exponents are degraded to suffixes. Thus, if m, n are integers > 0,

$$\alpha^m\alpha^n = \alpha^{m+n} = \alpha_{m+n}; \qquad \frac{\partial^r \alpha^m}{\partial \alpha^r} = \frac{m!}{(m-r)!}\alpha_{m-r};$$

$$(\alpha + \beta)^n = \sum_{r=0}^{n} \binom{n}{r} \alpha^{n-r}\beta^r = \sum_{r=0}^{n} \binom{n}{r} \alpha_{n-r}\beta_r.$$

These sufficiently illustrate the nature of the processes, which are fully discussed in the works cited.

4. If a is an ordinary, α an umbra, $a\alpha$ is by definition the umbra of the class $a\alpha_n$ $(n = 0, 1, 2, \cdots)$.

We define the equality $\alpha = \beta$ between umbræ α, β to mean either of the equivalent relations

$$\alpha^n = \beta^n, \qquad \alpha_n = \beta_n \quad (n = 0, 1, 2, \cdots).$$

The umbral multinomial theorem, which can be taken as the definition of $(a\alpha + b\beta + \cdots + c\gamma)^n$ for $a, b, \cdots, c$ ordinaries, $\alpha, \beta, \cdots, \gamma$ umbræ, and $n \geqq 0$ an integer, is

$$(a\alpha + b\beta + \cdots + c\gamma)^n = \sum \frac{n!}{f!\,g!\,\cdots\,h!} a^f b^g \cdots c^h \alpha_f \beta_g \cdots \gamma_h,$$

the Σ extending to all sets of integers $f, g, \cdots, h$ each $\geqq 0$ whose sum is n and whose number is equal to that of the umbræ. The right-hand side of this defines a polynomial δ_n in $a, b, \cdots, c$ and $\alpha_f, \beta_g, \cdots, \gamma_h$, for $f, g, \cdots, h$ as stated, which is uniquely determined when $a, b, \cdots, c, \alpha, \beta, \cdots, \gamma$ and n are given. Hence by the definition of equality we can write

$$\delta = a\alpha + b\beta + \cdots + c\gamma.$$

Suppose we have also $\delta = r\rho + s\sigma + \cdots + t\tau$. Then evidently the meaning of

$$a\alpha + b\beta + \cdots + c\gamma = r\rho + s\sigma + \cdots + t\tau$$

is uniquely defined.

From the foregoing definitions it follows immediately that

$$a\alpha + b\beta + c\gamma = a\alpha + c\gamma + b\beta = (a\alpha + b\beta) + c\gamma = a\alpha + (b\beta + c\gamma),$$

so that, *provided the umbræ are distinct*, the laws of addition and subtraction in U are precisely as in A.

The divergence between U, A is seen in the simplest instance. We do *not* have $\alpha + \alpha = 2\alpha$, for this would imply

$$2^n\alpha_n = \sum_{r=0}^{n} \binom{n}{r} \alpha_{n-r}\alpha_r,$$

which is in general false.

Suppose there are precisely n (>0) umbræ in the set $\alpha, \beta, \cdots, \gamma$, and precisely n ordinaries in the set $a, b, \cdots, c$, and let us write

$$\delta = a\alpha + b\beta + \cdots + c\gamma.$$

The result of replacing each of $\alpha, \beta, \cdots, \gamma$ by α in δ^m *after* degradation of all exponents of umbræ is defined to be the value of $(a\alpha + b\beta + \cdots + c\gamma)^m$. Hence if in any relation between umbræ a particular umbra occurs precisely n times, the n like umbræ are replaced by n distinct umbræ until after the degradation of exponents, when these n distinct umbræ are replaced by the original. In the particular case $a = b = \cdots = c = 1$ of the above, we write

$$n \cdot \alpha = \alpha + \alpha + \cdots + \alpha,$$

the α on the right being repeated precisely n times. Combining this with previous definitions, we have uniquely defined

$$n \cdot a\alpha + m \cdot b\beta + \cdots + s \cdot c\gamma$$

for $n, m, \cdots, s$ positive non-zero integers, $a, b, \cdots, c$ ordinaries, and $\alpha, \beta, \cdots, \gamma$ umbræ.

The symbol for zero, 0, is the symbol of an ordinary. The product 0α, written without the dot, is the umbra of the class each of whose elements is the ordinary zero, $0\alpha_i$ $(i = 0, 1, 2, \cdots)$.

From what precedes, *viz.*, from the special case of $(a\alpha + b\beta + \cdots)^m$ in which $a = 1$, $b = -1$, and each succeeding coefficient $= 0$, it follows that $\alpha - \alpha, \beta - \beta, \gamma - \gamma, \cdots$ are distinct, and we have shown how their values are to be calculated. With the distinction that the difference of two identical umbræ is never to be replaced by the symbol 0, subtraction in U is subject to all the laws of subtraction in A. For example,

$$\alpha - \alpha + \beta = \alpha + \beta - \alpha.$$

Combining results, we have now defined

$$n \cdot a\alpha + m \cdot b\beta + \cdots + s \cdot c\gamma$$

for $n, m, \cdots, s$ integers $\gtrless 0$; $a, b, \cdots, c$ ordinaries; $\alpha, \beta, \cdots, \gamma$ umbræ. Note that each of $n \cdot a\alpha, m \cdot b\beta, \cdots, s \cdot c\gamma$ is a single umbra.

5. Multiplication in U, with the exception of multiplication by $\alpha - \alpha$, is as in A. For, if we represent the umbral product of the umbræ α, β by $\alpha\beta$, and define $\alpha\beta$ to be the umbra of the class $\alpha_n\beta_n$ $(n = 0, 1, 2, \cdots)$, then $\alpha\beta = \beta\alpha$ and

$$(\alpha + \beta)(\gamma + \delta) = \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta = (\gamma + \delta)(\alpha + \beta),$$

as can be verified at once from the definition of equality. Again, if all the letters

are as at the end of § 4, and accented letters have similar meanings, it is evident that

$$(n\cdot a\alpha + m\cdot b\beta)(n'\cdot a'\alpha' + m'\cdot b'\beta')$$
$$= (n\cdot a\alpha)(n'\cdot a'\alpha') + (n\cdot a\alpha)(m'\cdot b'\beta') + (m\cdot b\beta)(n'\cdot a'\alpha') + (m\cdot b\beta)(m'\cdot b'\beta'),$$

where the separate terms $(n\cdot a\alpha)$, etc., are enclosed in ( ) to avoid a possible confusion, thus

$$(n\cdot a\alpha)(n'\cdot a'\alpha') \neq nn'\cdot aa'\alpha\alpha'.$$

That is, $n\cdot a\alpha$, or any similar term, is treated as the *single* umbra which it is.

The treatment of the excepted case, multiplication by $\alpha - \alpha$, is referred by means of

$$(\alpha - \alpha)\beta = \alpha\beta - \alpha\beta$$

to § 4, so that this case also follows the same laws as above. The distinction here is that $\alpha\beta - \alpha\beta$ is not zero, as it would be in A.

6. It remains to say a word about the way in which U is applied. Assume for a moment that between the umbræ $\alpha, \beta, \cdots, \gamma$ and δ we have found the relation

$$\delta = \alpha + \beta + \cdots + \gamma,$$

and let x be a parametric ordinary, *viz.*, a parameter whose range of values is the range of ordinaries, such that $x^n \neq \xi_n$ $(n = 0, 1, 2, \cdots)$ for $\xi = \alpha, \beta, \cdots, \gamma, \delta$. Let $f(x)$ be a polynomial in x. Then, from the assumed umbral equality, it follows that

$$f(x + \delta) = f(x + \alpha + \beta + \cdots + \gamma),$$

and there is an extension of this, which we need not discuss here, to the case in which $f(x)$ is any function possessing a power series expansion in x.

To prove the polynomial relation it suffices to observe that for all integers $n \geqq 0$ we have

$$(x + \delta)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r}\delta^r,$$
$$= \sum_{r=0}^{n} \binom{n}{r} x^{n-r}(\alpha + \beta + \cdots + \gamma)^{n-r},$$

by means of the umbral equality, and hence

$$(x + \delta)^n = (x + \alpha + \beta + \cdots + \gamma)^n,$$

which might have been written down immediately from the umbral equality, as follows. We can consider x as an umbra, *viz.*, x_n is here $\equiv x^n$, since x itself is an ordinary, and x, quâ umbra, is the umbra of the class of ordinaries x^n $(n = 0, 1,$

2, $\cdots$). Since by hypothesis the umbræ $x, \alpha, \beta, \cdots, \gamma, \delta$ are distinct, it follows from $\delta = \alpha + \beta + \cdots + \gamma$ by the properties of addition of umbræ that $x + \delta = x + \alpha + \beta + \cdots + \gamma$, and this is equivalent to the relation stated.

Hence from any umbral equality may be inferred immediately an equality between polynomials (or functions, as mentioned) whose arguments contain, in addition to the umbræ concerned, a parameter x. By giving to x special values, and by choosing for the polynomial (or function) particular forms, we obtain an unlimited number of algebraic relations between the $\alpha_n, \beta_n, \cdots, \gamma_n, \delta_n$ for different n's.

To obtain the initial umbral equality we proceed from the *generators* of $\alpha_n, \beta_n, \cdots$. The generator of α_n is by definition $e^{\alpha y}$, where y is a parametric ordinary, *viz.*,

$$e^{\alpha y} \equiv \alpha_0 + \alpha_1 y + \alpha_2 \frac{y^2}{2!} + \cdots.$$

Suppose that a certain function $F(y)$ of y has a MacLaurin expansion, so that we can write

$$F(y) = e^{\alpha y},$$

where α_n is the coefficient of $y^n/n!$ in the MacLaurin series for $F(y)$. Suppose further that in the same way we have

$$G(y) = e^{\beta y}, \cdots, \qquad H(y) = e^{\gamma y}, \qquad K(y) = e^{\delta y},$$

and that there exists the (ordinary) identity in y,

$$K(y) = F(y)G(y) \cdots H(y).$$

Then it follows that

$$e^{\delta y} = e^{(\alpha+\beta+\cdots+\gamma)y},$$

whence

$$\delta = \alpha + \beta + \cdots + \gamma.$$

The actual, fruitful process is somewhat different from this. We start from any ordinary function of y, and resolve it in a given domain into its irreducible factors. The function and each of its factors are then expanded and the expansion written in exponential umbral form. By combining the factors in various ways different umbral equalities between members of a set of related functions are obtained as above outlined.

To get interesting results, we proceed as follows. Let

$$I(a, x, y, \cdots, z) = 0$$

be an identity between ordinaries $a, x, y, \cdots, z$. Replace $a, x, y, \cdots, z$ by θ, $e^{p\chi\theta}$, $e^{q\chi\theta}$, $\cdots$, $e^{r\chi\theta}$ respectively, where $\theta, p, q, \cdots, r$ are ordinaries. Call the

transformed identity I'. It is equivalent to an umbral identity

$$\alpha = 0,$$

which is obtained by putting the left-hand member of I' into its generator form $e^{a\theta}$. From then on the procedure is as previously sketched. In this way we use the simplest ordinary identities as the points of departure for reading off general umbral polynomial or functional relations between classes of numbers or functions. A powerful feature of this isomorphism between A, U is its great suggestiveness as to new numbers and functions worthy of investigation.

We have omitted umbral division α/β, because its definition and discussion appear to be of but slight utility.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS.

I. PI AND THE FACTORS OF $x^2 + 1$.

BY A. A. BENNETT, University of Texas.

The three term Diophantine arccotangent relation

$$\operatorname{arccot} x = \operatorname{arccot} y + \operatorname{arccot} z \tag{1}$$

is of importance [1] in the computation of π. On the other hand the problem of determining integers to satisfy (1) is equivalent to the problem of finding the numerical factors of $x^2 + 1$. Indeed (1) may be written in the form

$$\operatorname{arccot} x = \operatorname{arccot} (x + u) + \operatorname{arccot} (x + v), \qquad uv = x^2 + 1, \tag{2}$$

as may be seen from the equation $z = (xy + 1)/(y - x)$, to which (1) may be reduced, if y is replaced by $x + u$.

A table of the values of $x^2 + 1$ may readily be constructed, for successive integral values of x, and these values of $x^2 + 1$ may be tested for prime and composite factors in the usual ways. What is of particular interest is that, as often happens, such a table can be checked with great ease although at first sight it might be natural to suppose that a composite number with only large prime factors might be mistaken for a prime by some slip in computation. We have in the first place the following theorem.

A necessary and sufficient condition that a given integer, u, be a factor of

[1] *Cf.* Hobson, *Plane Trigonometry*, p. 310.

an integer of the form $x^2 + 1$ is that u contains no factor of the form $4k + 3$, and that u does not contain 4 as a factor. However u may be even.

The proof of this is merely an exercise in the first principles of quadratic residues. The hypothesis may be stated in familiar technical language in the form $x^2 \equiv -1$, mod u. By the usual rules, we see that since -1 is not congruent to 1, mod 4, u cannot contain 2 to a power higher than the first, and -1 must be a quadratic residue of each prime factor of u. For -1 to be a quadratic residue of an odd prime, p, it is necessary and sufficient that $(-1)^{(p-1)/2} \equiv 1$, mod p. Thus p cannot be of the form $4k + 3$ but must be of the form $4k + 1$. Incidentally, any product of such factors is again of this same form. The general theory shows that these conditions are sufficient as well as necessary.

In the second place we have the following as a consequence of the general propositions of elementary number theory, namely, the assurance that for any given odd prime of the form $4k + 1$ there will be two and only two distinct values of r, such that $0 < r < u$, while $r^2 + 1$ is divisible by u. The actual determination of these values of r for a given u may be effected without much difficulty for moderate values of u. It is perhaps best explained by reference to an example. Take for instance the prime, $853 = 4(213) + 1$, as u. Express u as the sum of two squares, which by the general theory is always uniquely possible for u of this form. This is not difficult for moderate values of u by the mere use of a table of squares, trying successively in decreasing order the squares which are less than u. Thus 853 is quickly found to be expressible as $23^2 + 18^2$. Use the smaller of these two numbers (not squared) as partial divisor, writing $23/18$ as $1 + (5/18)$. Then as a congruence we have $18^2(1 + [1 + (5/18)]^2) \equiv 0$, mod 853, from which the external factor, 18^2, may be dropped. The problem now is to express $5/18$ as congruent to an integer with respect to the given modulus. Now 853 is itself congruent to zero, that is, $18[47 + (7/18)]$ is congruent to zero, and hence $7/18 \equiv -47$, mod 853. Hence trying successive multiples of $7/18$, we find that $49/18 = 3 - (5/18) \equiv -7.47$, mod 853, or $1 + (5/18) \equiv 4 + 7.47 = 333$, mod 853. The two values of r may be thought of as differing only in sign, or if positive values are desired, they are such that their sum is the given u. Here the values are 333, and $853 - 333 = 520$. As a check we have $333^2 = 110889$, and $333^2 + 1 = 110890 = 853.130$. Similarly $520^2 + 1 = 270401 = 853.317$.

We have then the following facts to be used in checking a table of the factors of $x^2 + 1$,

(i) Each of the numbers 1, 2, 5, 10, 13, 17, 25, 26, 29, $\cdots$ is a factor of some $x^2 + 1$, and if one of these, say u, is a factor of a given $x^2 + 1$, it is a factor of each successive $(x + nu)^2 + 1$. The possible values of x smaller than u are also capable of fairly simple direct computation, as has been explained for the case in which u is prime.

(ii) Each factor of $x^2 + 1$ is in this sequence 1, 2, 5, 10, 13, $\cdots$, which consists of those integers which contain neither 4 nor any factor of the form $4k + 3$.

(iii) Whenever a u is a factor of an $x^2 + 1$, for an x less than u, another

factor, v, is obtained, where v is greater than x, which in turn serves as a u for later expressions.

(iv) Each u appears first as a factor of $x^2 + 1$, for an x less than u and indeed for at least two distinct such cases. It can be overlooked only if the other factor, say v, is ignored, and hence for v less than x, if for some previous cases, namely, $x - v$, $x - 2v$, $\cdots$, this same factor v is ignored, and so on in infinite descent.

Thus the table when once started may be continued largely by its own momentum, with next to no computation, and may be rechecked in several ways at any stage.

The following table gives the values of x, u, v, for $|x| \leq 40$. In each case the smaller of the factors u, v is placed over the larger, and positive signs only are given.[1]

1.	1			11.	1	2			21.	1	2	13	17	31.	1	2	13	26	
	2				122	61				442	221	34	26		962	481	74	37	
2.	1			12.	1	5			22.	1	5			32.	1	5	25		
	5				145	29				485	97				1025	205	41		
3.	1	2		13.	1	2	5	10	23.	1	2	5	10	33.	1	2	5	10	
	10	5			170	85	34	17		530	265	106	53		1090	545	218	109	
4.	1			14.	1				24.	1				34.	1	13			
	17				197					577					1157	89			
5.	1	2		15.	1	2			25.	1	2			35.	1	2			
	26	13			226	113				626	313				1226	613			
6.	1			16.	1				26.	1				36.	1				
	37				257					677					1297				
7.	1	2	5	17.	1	2	5	10	27.	1	2	5	10	37.	1	2	5	10	
	50	25	10		290	145	58	29		730	365	146	73		1370	685	274	137	
8.	1	5		18.	1	5	13		28.	1	5			38.	1	5	17		
	65	13			325	65	25			785	157				1445	289	85		
9.	1	2		19.	1	2			29.	1	2			39.	1	2			
	82	41			362	181				842	421				1522	761			
10.	1			20.	1				30.	1	17			40.	1				
	101				401					901	53				1601				

II. On the Use of the Calculating Machine for Cube and Fifth Roots.

By Derrick Henry Lehmer, Berkeley, Calif.

The extraction of roots of higher degree than the second by means of the calculating machine is difficult for any but the most experienced computers and is at best a slow process. The subjoined table has been prepared to make this process both rapid and accurate for cube and fifth roots. It covers the range of numbers from *1* to *10,000,000,000* and presents a remarkable condensation when compared with other available tables of much smaller range.

It will be seen that the table has for arguments the first nine integers together with numbers of the form $1.0^n p$. Its use therefore depends on the decomposition

[1] To illustrate its application to (1), take 13, and the corresponding factors 5 and 34 of $13^2 + 1$. Then we may infer that

$$\operatorname{arccot} 13 = \operatorname{arccot} (13 + 5) + \operatorname{arccot} (13 + 34),$$

$$\operatorname{arccot} 13 = \operatorname{arccot} (13 - 5) + \operatorname{arccot} (13 - 34).$$

of a number into factors of this particular type. This decomposition, which has been made use of for the computation of logarithms,[1] may be effected in the following manner which is well adapted to the computing machine:

If the given number lies between one and ten, divide first by the greatest integer in it. Divide the quotient by the greatest number of the form $1.0^n p$ that is less than the quotient. Proceed in the same way with this quotient and continue the process until a quotient is reached that is sensibly unity. If the given number does not lie between one and ten, it can be made to do so by division by an appropriate power of ten.

In practice one finds that only half the above process needs to be carried out, the last factors being obtained by inspection.

As an illustration let it be required to decompose the number 432.7699836. The number 4.327699836 is first divided by 4. The quotient, 1.081924959, is then divided by the number 1.08. The quotient, $1.0^2$1782369, is in turn divided by $1.0^2$1. The quotient, $1.0^3$781587, is now divided by $1.0^3$7. The quotient, $1.0^4$81530, might now be divided by the number $1.0^4$8 and the process continued, but the remaining factors are seen without division to be $1.0^5$1, $1.0^6$5, and $1.0^7$3. The given number, 432.7699836, is thus the product of the following numbers: 10^2, 4, 1.08, $1.0^2$1, $1.0^3$7, $1.0^4$8, $1.0^5$1, $1.0^6$5, $1.0^7$3.

To apply this decomposition to the extraction of roots, we observe that the root of any given number will be the product of the corresponding roots of the factors. The subjoined table gives the cube and fifth roots of all numbers of the above form to ten significant figures. The following examples show the method of using the table.

Example 1. Required the cube root of 34625.00589. Decomposing this number according to the method indicated above, we have

$$34625.00589 = 10^1 \times 3 \times 1.1 \times 1.04 \times 1.0^2 8 \times 1.0^3 8 \times 1.0^4 8 \times 1.0^7 1 \times 1.0^8 3.$$

Multiplying together the values found opposite these factors in the cube root column, we get the required root, 32.59342142.

Example 2. Required the fifth root of .0005734267268. We have, decomposing the number 5.734267268,

$$5.734267268 = 5 \times 1.1 \times 1.04 \times 1.0^2 2 \times 1.0^3 4 \times 1.0^4 9 \times 1.0^5 3 \times 1.0^6 2 \times 1.0^7 5 \times 1.0^8 3.$$

The product of the numbers opposite these factors in the fifth root column is 1.418063200 which is the fifth root of 5.734267268. To get the root of the required number we multiply this result by $\sqrt[5]{10}/10$. The final result is .2247478713.

After so many successive multiplications it is quite possible that the last digit may be erroneous. If ten-figure accuracy is required, the following correction can easily be made: For a cube root, $c = e/(3r^2)$, where e is the difference between the given number and the cube of the tentative root r. The corresponding correction for the fifth root is $c = e/(5r^4)$. Thus in example 2 the error in the last digit is as much as 4. Applying the correction indicated, the true value is .2247478709.

[1] See last column of article on *Logarithms* by J. W. L. Glaisher in the *Encyc. Brit.*

TABLE OF CUBE AND FIFTH ROOTS.

n	$\sqrt[3]{n}$	$\sqrt[5]{n}$	n	$\sqrt[3]{n}$	$\sqrt[5]{n}$
1	1.0000 00000	1.0000 00000	$1.0^{1}1$	$1.0^{4}03333$	$1.0^{4}02000$
2	1.2599 21050	1.1486 98355	$1.0^{1}2$	$1.0^{4}06667$	$1.0^{4}04000$
3	1.4422 49570	1.2457 30940	$1.0^{1}3$	$1.0^{4}10000$	$1.0^{4}06000$
4	1.5874 01052	1.3195 07911	$1.0^{1}4$	$1.0^{4}13333$	$1.0^{4}08000$
5	1.7099 75947	1.3797 29662	$1.0^{1}5$	$1.0^{4}16666$	$1.0^{4}10000$
6	1.8171 20593	1.4309 69081	$1.0^{1}6$	$1.0^{4}20000$	$1.0^{4}12000$
7	1.9129 31183	1.4757 73162	$1.0^{1}7$	$1.0^{4}23333$	$1.0^{4}14000$
8	2.0000 00000	1.5157 16567	$1.0^{1}8$	$1.0^{4}26666$	$1.0^{4}15999$
9	2.0800 83823	1.5518 45574	$1.0^{1}9$	$1.0^{4}29999$	$1.0^{4}17999$
1.1	1.0322 80115	1.0192 44877	$1.0^{2}1$	$1.0^{3}0333$	$1.0^{3}0200$
1.2	1.0626 58569	1.0371 37289	$1.0^{2}2$	$1.0^{3}0667$	$1.0^{3}0400$
1.3	1.0913 92883	1.0538 73952	$1.0^{2}3$	$1.0^{3}1000$	$1.0^{3}0600$
1.4	1.1186 88942	1.0696 10376	$1.0^{2}4$	$1.0^{3}1333$	$1.0^{3}0800$
1.5	1.1447 14243	1.0844 71771	$1.0^{2}5$	$1.0^{3}1667$	$1.0^{3}1000$
1.6	1.1696 07095	1.0985 60543	$1.0^{2}6$	$1.0^{3}2000$	$1.0^{3}1200$
1.7	1.1934 83192	1.1119 61586	$1.0^{2}7$	$1.0^{3}2333$	$1.0^{3}1400$
1.8	1.2164 40399	1.1247 46113	$1.0^{2}8$	$1.0^{3}2667$	$1.0^{3}1600$
1.9	1.2385 62330	1.1369 74489	$1.0^{2}9$	$1.0^{3}3000$	$1.0^{3}1800$
1.01	1.0033 22284	1.0019 92048	$1.0^{3}1$	$1.0^{2}033$	$1.0^{2}020$
1.02	1.0066 27709	1.0039 68378	$1.0^{3}2$	$1.0^{2}067$	$1.0^{2}040$
1.03	1.0099 01634	1.0059 29269	$1.0^{3}3$	$1.0^{2}100$	$1.0^{2}060$
1.04	1.0131 59404	1.0078 74989	$1.0^{3}4$	$1.0^{2}133$	$1.0^{2}080$
1.05	1.0163 96357	1.0098 05798	$1.0^{3}5$	$1.0^{2}167$	$1.0^{2}100$
1.06	1.0196 12822	1.0117 21952	$1.0^{3}6$	$1.0^{2}200$	$1.0^{2}120$
1.07	1.0228 09122	1.0136 23698	$1.0^{3}7$	$1.0^{2}233$	$1.0^{2}140$
1.08	1.0259 85568	1.0155 11278	$1.0^{3}8$	$1.0^{2}267$	$1.0^{2}160$
1.09	1.0291 42467	1.0173 84928	$1.0^{3}9$	$1.0^{2}300$	$1.0^{2}180$
$1.0^{2}1$	1.0003 33222	1.0001 99920	$1.0^{7}1$	$1.0^{7}03$	$1.0^{7}02$
$1.0^{2}2$	1.0006 66223	1.0003 99680	$1.0^{7}2$	$1.0^{7}07$	$1.0^{7}04$
$1.0^{2}3$	1.0009 99002	1.0005 99281	$1.0^{7}3$	$1.0^{7}10$	$1.0^{7}06$
$1.0^{2}4$	1.0013 31559	1.0007 98723	$1.0^{7}4$	$1.0^{7}13$	$1.0^{7}08$
$1.0^{2}5$	1.0016 63897	1.0009 98006	$1.0^{7}5$	$1.0^{7}17$	$1.0^{7}10$
$1.0^{2}6$	1.0019 96013	1.0011 97130	$1.0^{7}6$	$1.0^{7}20$	$1.0^{7}12$
$1.0^{2}7$	1.0023 27910	1.0013 96097	$1.0^{7}7$	$1.0^{7}23$	$1.0^{7}14$
$1.0^{2}8$	1.0026 59587	1.0015 94905	$1.0^{7}8$	$1.0^{7}27$	$1.0^{7}16$
$1.0^{2}9$	1.0029 91045	1.0017 93555	$1.0^{7}9$	$1.0^{7}30$	$1.0^{7}18$
$1.0^{3}1$	1.0000 33332	1.0000 19999	$1.0^{8}1$	$1.0^{8}0$	$1.0^{8}0$
$1.0^{3}2$	1.0000 66662	1.0000 39997	$1.0^{8}2$	$1.0^{8}1$	$1.0^{8}0$
$1.0^{3}3$	1.0000 99990	1.0000 59993	$1.0^{8}3$	$1.0^{8}1$	$1.0^{8}1$
$1.0^{3}4$	1.0001 33316	1.0000 79987	$1.0^{8}4$	$1.0^{8}1$	$1.0^{8}1$
$1.0^{3}5$	1.0001 66639	1.0000 99980	$1.0^{8}5$	$1.0^{8}2$	$1.0^{8}1$
$1.0^{3}6$	1.0001 99960	1.0001 19971	$1.0^{8}6$	$1.0^{8}2$	$1.0^{8}1$
$1.0^{3}7$	1.0002 33279	1.0001 39961	$1.0^{8}7$	$1.0^{8}2$	$1.0^{8}1$
$1.0^{3}8$	1.0002 66596	1.0001 59949	$1.0^{8}8$	$1.0^{8}3$	$1.0^{8}2$
$1.0^{3}9$	1.0002 99910	1.0001 79935	$1.0^{8}9$	$1.0^{8}3$	$1.0^{8}2$

$\sqrt[3]{10} = 2.1544\ 34690$ $\qquad$ $\sqrt[3]{100} = 2.5118\ 86432$

$\sqrt[3]{100} = 4.6415\ 88834$ $\qquad$ $\sqrt[5]{1000} = 3.9810\ 71705$

$\sqrt[5]{10} = 1.5848\ 93193$ $\qquad$ $\sqrt[5]{10000} = 6.3095\ 73445$

RECENT PUBLICATIONS.

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

THE FIRST CARUS MONOGRAPH.

Calculus of Variations. By G. A. BLISS. Chicago, The Open Court Publishing Company, 1925. xiii + 189 pages. Price $2.00 post-paid to non-members of the Association.

This review might very properly begin with the quotation of the first paragraph of the author's preface which explains the origin and purpose of these monographs. "This book is the first of a series of monographs on mathematical subjects which are to be published under the auspices of the Mathematical Association of America and whose publication has been made possible by a very generous gift to the Association by Mrs. Mary Hegeler Carus as trustee of the Edward C. Hegeler Trust Fund. The purpose of the monographs is to make the essential features of various mathematical theories accessible and attractive to as many persons as possible who have an interest in mathematics but who may not be specialists in the particular theory presented, a purpose which Mrs. Carus has very appropriately described to be the 'diffusion of mathematical and formal thought as contributory to exact knowledge and clear thinking not only for mathematicians and teachers of mathematics but also for other scientists and the public at large.'"

The principal aim of this review is to give an appraisal of this work as to the extent to which it has fulfilled the purposes of these monographs, leaving to others to give a detailed description and criticism of the subjects discussed. I might say at the outset that the publication committee could have made no wiser choice for the subject of the first monograph than that of the calculus of variations. The beautiful geometrical and mechanical properties of curves and surfaces which unfold themselves so naturally in this subject are sure to arrest the attention of the reader who is even only mildly interested in mathematics. Coupled with this inherent beauty of the subject, we have here an exposition by a man whose ability both as a teacher and scholar is of such outstanding character as to insure the success of any mathematical work, let alone a subject with which he has such an intimate acquaintance and in which he has such a deep interest. Those of us who have always regarded mathematics merely as a science find in these pages an artistic structure and a beauty of technique that lead us to think that mathematics in this country is at last taking on the form of an art also.

The book opens with an introductory chapter giving a statement and historical sketch of some of the typical problems of the calculus of variations. This chapter is so clear and vivid that it must surely arouse the interest and curiosity of the reader. For those who have no acquaintance with this subject it may not

be amiss at this place to give a brief statement of one of the most interesting of these problems, viz., the brachistochrone problem which was first proposed by John Bernoulli in 1696. For some reason a rivalry had sprung up between him and his older brother James and this problem was proposed as a sort of a challenge. It is required to pick out of all the possible paths joining two points in a vertical plane that one down which a particle will fall in the shortest time. It is clear that this problem is one which is quite distinct from the maxima and minima problems solved in the differential calculus. In fact we have here to find not merely a value, or values, of a variable which will give an assigned function the largest or smallest value, but a whole curve or function which, when substituted in the integrand of a certain definite integral, will give the latter a value smaller than would be obtained by the substitution of any other function. The brothers both solved the problem in 1697; and they found much to their surprise that the curve down which a particle will fall in the least time is the same curve which will be generated by a particle on the rim of a wheel as the wheel rolls along a straight line, and more surprising still it is the same curve on which a particle starting at rest will fall to the lowest point in the same time no matter at what point of the curve the particle is started. In other words, a pendulum constrained to move along a cycloid, as this curve is called, would have a period independent of the arc of swing, a property not enjoyed by the so-called simple pendulum. We can feel with John Bernoulli when he says (see page 54), "With justice we admire Huygens because he first discovered that a heavy particle falls on a cycloid in the same time always, no matter what the starting point may be. But you will be petrified with astonishment when I say that exactly this same cycloid, the tautochrone of Huygens, is the brachistochrone which we are seeking."

In the next three chapters this and two other problems are taken up in great detail with a view to explaining and illustrating the modern methods of the calculus of variations. These modern methods of studying this subject are here presented for the first time in a form which it is possible for the intelligent layman with a thorough training in advanced calculus to understand. I do not mean to imply that the reader will always find it easy to follow every statement at a first glance. In fact I doubt if any one but a specialist could read everything in the book that way. But I am sure that it is possible for a reader with a good training in elementary mathematics to verify every statement made in this book, provided he has sufficient patience and concentration. If we recall that mathematicians had been working on this subject for over 300 years before it was finally put on a rigorous foundation by Weierstrass, we can see why a beginner should not expect to understand the calculus of variations in all its aspects at a first reading. In fact, I would suggest to the beginner to omit on a first reading all the difficult proofs in the book and come back to them only after he has seen the underlying general ideas of the subject. This, by the way, is I think the best and most efficient method of reading all mathematics, even technical articles. But, in spite of the inherent difficulty of the subject, Professor Bliss

has given us an exposition which I think will be considered a model for many years to come. Whenever he needs to use some result from an advanced portion of mathematics, as for example the theory of implicit functions, he either substitutes for it a clear geometric proof (see for example the proof that a cycloid can always be drawn through two points, page 56) or he gives a proof using nothing more than what one should know after a thorough course in calculus.

The book ends with a chapter on the general theory and we have to marvel that any one can in the space of fifty pages give such a clear and complete treatment of what is known as the *simplest problem* of the calculus of variations. Yet we actually find that the necessary conditions as well as the sufficient conditions for a minimum or maximum are stated and proved with all the rigor that one could desire. This chapter should be excellent reading for one who has already had some acquaintance with the calculus of variations, and even the specialist will find here a form which is not lacking in novelty. The chapter closes with an interesting historical account of the whole subject.

It may be seen from this brief description that this is an interesting and useful book for a large group of readers whose mathematical equipment varies all the way from a course in the differential and integral calculus to one in the calculus of variations. In fact, I should think it would make an excellent introduction for a course in this subject if not as a text, at least for collateral reading.

I. A. BARNETT.

THREE BOOKS ON INFINITE SERIES.

1. *Theorie und Anwendungen der unendlichen Reihen.* By K. KNOPP. Berlin, Julius Springer, 1922. 474 pages. Price 28 gold marks.

This is a text on fundamental theory. The subject matter treated and results obtained although generally well known are put forward here in a particularly clear and accessible manner. There is a treatment of the theory of aggregates and in particular of the number system. The treatment of convergence and divergence of sequences and series in general is good. A large number of theorems are clearly stated and proved. The comments of the author are scholarly. At times there seems to be quite needless repetition as many theorems are proved which appear as special cases of more general theorems proved later, and several statements of the same theorem will frequently be given where the change is merely a matter of wording. Special theories such as Dirichlet Series, summation of series, development of functions in series, etc., etc., are hurriedly and inadequately treated. Had they been treated with the thoroughness of the convergence and divergence theory, we should certainly have a very fine work on series. However, taken as a whole this is probably the best book on infinite series on the market at present.

2. *Elements of the Theory of Infinite Processes.* By L. L. SMAIL. New York, McGraw-Hill Book Co., 1923. 339 pages. Price $3.50.

This book is not unlike the preceding. It is intended for a text for advanced undergraduates or first year graduate students at an American university. As

the title indicates, the author does not limit himself to series but studies infinite products, continued fractions, integrals and determinants also. The whole is preceded by a study of the theory of aggregates. Theorems are well stated and proved. Infinite series naturally are the chief concern of the author. However, special types are not treated with sufficient fullness to serve as more than the merest introduction. The book is somewhat more elementary than that of Knopp and as a result, so far as the same topics are covered, less complete. But there is the advantage to the American of the English Language. As Bromwich's *Theory of Infinite Series* is now practically unobtainable, it fills a decided want and should be regarded as a worthwhile addition to American mathematical literature.

3. *An Introduction to the Operations with Series.* By I. J. SCHWATT. Philadelphia, Press of the University of Pennsylvania, 1924. 287 pages. $5.00 net.

This book is unlike any other thing known to the reviewer. It partakes as much of the nature of a table as of a text or treatise and whereas it is certainly a valuable book to have it is not a book that any one is likely to read in completeness. The obtaining of closed formulas for the sums of both finite and infinite series seems to be the principal end of the author although in many instances the reverse problem is handled. Formulas are obtained in large number and with but little comment. It is believed that many of his results are new and, although no effort at verification has been made, this seems extremely likely judging from their number and complexity. For reference the book is well worth having. It is a witness to the ingenuity and algebraic skill of its author. The book is handled by D. Appleton & Co., New York City.

TOMLINSON FORT.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

AMERICAN JOURNAL OF MATHEMATICS, volume 47, no. 2, April, 1925: "Some properties of the exponential mean" by J. Dale, 71–90; "The eliminant of a net of curves" by F. Morley, 91–97; "On an equation of planar motion" by F. Morley, 98–100; "An extension of the problem of the elastic bar" by H. T. Davis, 101–120; "On the Sylow subgroups of the symmetric and alternating groups" by L. Weisner, 121–124; "On certain theorems regarding summable series and their application to the double and triple Fourier's series" by G. M. Merriman, 125–139; "On the power characters of units in a cyclotomic field" by H. S. Vandiver, 140–147.

ANNALS OF MATHEMATICS, volume 26, nos. 1 and 2, September–December, 1924: "Contact transformations linear in x, y, z; applications to equilong transformations" by B. H. Brown, 1–7; "On the trigonometric representation of an ill-defined function" by D. Jackson, 8–20; "An irregular boundary value and expression problem" by L. E. Ward, 21–36; "Maximal cuspidal curves" by T. R. Hollcroft, 37–46; "Plane cubics associated with the quadrangle-quadrilateral configuration" by B. M. Turner, 47–58; "On Pellet's theorem concerning the roots of a polynomial" by J. L. Walsh, 59–64; "The setting of a proposition" by P. J. Daniell, 65–78; "Theta functions and arithmetic" by E. T. Bell, 79–87; "A new type of criteria for the first case of Fermat's last theorem" by H. S. Vandiver, 88–94; "Algebraic functions and their divisors"

by G. A. Bliss, 95–124; "An extension of the definition of the Green's function in one dimension" by W. M. Whyburn, 125–130; "Real representations of analytic complex curves" by W. C. Graustein, 131–143; "Note on Dirichlet series with complex exponents" by J. F. Ritt, 144; "On the order of an analytic function at a singular point" by M. H. Stone, 145–154; "Representations of integers in certain binary, ternary, quaternary, and quinary quadratic forms and allied class number relations" by E. T. Bell, 155–164.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 31, nos. 1–2, January–February, 1925: "A theorem on simple algebras" by J. H. M. Wedderburn, 11–13; "Note on a class of harmonic functions" by G. C. Evans, 14–16; "Three theorems on normal orthogonal sets" by M. H. Stone, 17–20; "On the complete independence of the functional equations of involution" by C. C. MacDuffee, 21–26; "The frequency law of a function of one variable" by E. L. Dodd, 27–31; "On the accessibility of an arc from its complement in space of three dimensions" by C. Kuratowski, 32; "On sets of three consecutive integers which are quadratic residues of primes" by H. S. Vandiver, 33–38; "Normal congruences of curves in Riemann space" by H. Levy, 39–41; "Limits for actual double points of space curves" by T. R. Hollcroft, 42–55; "Some mean-value theorems connected with Cotes' method of mechanical quadrature" by D. V. Widder, 56–62; "The geometry of frequency functions" by D. Jackson, 63–73. Nos. 3–4, March–April, 1925: "Remarks on the foundations of geometry" by O. Veblen, 121–141; "Ricci's coefficients of rotation" by H. Levy, 142–144; "Sur les valeurs asymptotiques des coefficients de Cotes" by J. Ouspensky, 145–156; "Functions with essential singularity" by P. Franklin, 157–162; "The calculus of variations" by L. Tonelli, 163–172.

JOURNAL OF MATHEMATICS AND PHYSICS, M. I. T., volume 4, no. 3, May, 1925: "The Weierstrass approximation theorem" by P. Franklin, 148–152; "The solution of a difference equation by trigonometric integrals" by N. Wiener, 153–163; "Note on a method of evaluating the complex roots of sixth and higher order equations" by L. F. Woodruff, 164–166; "Minimal varieties of two and three dimensions whose element of arc is a perfect square" by C. L. E. Moore, 167–178; "The multiple complement of one or more polyadics" by F. L. Hitchcock and L. H. Rice, 179–187; "Note on De Sitter's universe" by G. Lemaitre, 188–192.

JOURNAL FÜR DIE REINE UND ANGEWANDTE MATHEMATIK, volume 154, no. 2, January, 1925: "The number e in $k(p)$" by G. E. Wahlin, 110–113.

L'ENSEIGNEMENT MATHEMATIQUE, volume 24, nos. 1–2–3, March, 1925: "Histoire de cinq concepts fondamentaux des mathematiques" by G. A. Miller, 59–69.

MATHEMATISCHE ANNALEN, volume 94, nos. 1–2, April, 1925: "Einstein spaces which are mapped conformally on each other" by H. W. Brinkmann, 119–145.

MATHEMATISCHE ZEITSCHRIFT, volume 23, nos. 1–2, March, 1925: "Some problems in 'Partitio numerorum' (VI): Further researches in Waring's problem" by G. H. Hardy and J. E. Littlewood, 1–37.

MESSENGER OF MATHEMATICS, volume 54, nos. 9–10, January–February, 1925: "The stability of electrons and protons" by H. Bateman, 142–149.

PHILOSOPHICAL MAGAZINE, volume 49, no. 293, May, 1925: "The motion of an electric charge" by A. Bramley, 912–923.

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, volume 11, no. 4, April, 1925: "On the projective and equi-projective geometries of paths" by T. Y. Thomas, 199–203; "Projective normal co-ordinates for the geometry of paths" by O. Veblen and J. M. Thomas, 204–206; "Note on the projective geometry of paths" by J. M. Thomas, 207–208. No. 5, May, 1925: "Linear connections of a space which are determined by simply transitive continuous groups" by L. P. Eisenhart, 246–249; "Note on a theorem by H. Kneser" by J. W. Alexander, 250–251; "Associated types of linear connection" by L. Ingold, 252–256; "Conformal correspondence of Riemann spaces" by J. M. Thomas, 257–259. No. 6, June, 1925: "A note on the abundance of differential combinants in a fundamental system" by O. E. Glenn, 281–283; "On the application of Borel's method to the summation of Fourier's Series" by C. N. Moore, 284–286; "The intersection of complexes of manifolds" by S. Lefschetz, 287–289; "Continuous transformations of manifolds" by S. Lefschetz, 290–291; "Laws of Reciprocity and the first case of Fermat's last theorem" by H. S. Vandiver, 292–297.

SCHOOL SCIENCE AND MATHEMATICS, volume 25, no. 4, April, 1925: "Arithmetic in the junior high school" by L. W. Colwell, 363–369; "The three-step method in teaching Geometry" by F. L. Abbott, 409–411. No. 5, May, 1925: "Positive and negative numbers" by H. C. Christofferson, 507–514; "Speed and scholarship vs. arithmetical accuracy" by W. W. Ludeman, 522–524. No. 6, June, 1925: "Geometry in the junior high school" by J. T. Johnson, 611–617.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 27, no. 1, January, 1925: "On normal forms of differential equations" by W. F. Osgood, 1–14; "Congruence with constant absolute invariants" by H. L. Olson, 15–42; "On the prime divisors of the cyclotomic functions" by C. M. Huber, 43–48; "On the roots of the Riemann zeta function" by J. I. Hutchinson, 49–60; "A generalization of the Riemannian line element" by J. L. Synge, 61–67; "Elementary functions and their inverses" by J. F. Ritt, 68–90; "Analytic transformations of everywhere dense point sets" by P. Franklin, 91–100; "An algebraic solution of the Einstein equations" by E. Kasner, 101–105; "Electrodynamics in the general relativity theory" by G. Y. Rainich, 106–136. No. 2, April, 1925: "The subgroup composed of the substitutions which omit a letter of a transitive group" by G. A. Miller, 137–145; "On the closeness of approach of complex rational fractional to a complex irrational number" by L. R. Ford, 146–154; "Solutions of the Einstein equations involving functions of only one variable" by E. Kasner, 155–162; "A general theory of linear sets" by M. H. Ingraham, 163–196; "On the representation of a certain fundamental law of probability" by H. L. Rietz, 197–212; "The group of motions of an Einstein space" by J. Eiesland, 213–245; "A generalization of Levi-Civita's parallelism and the Frenet formulas" by J. H. Taylor, 246–264.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3144. Proposed by JOHN BIGGERSTAFF, University of Washington.

Find the point from which the sum of the distances to two given straight lines and the distance to a given point is a minimum.

[Joseph Bertrand.]

3145. Proposed by W. J. SIDIS, New York City.

Prove that if a number of n digits, expressed in the scale of r, is divisible by any factor of $r^n - 1$, that divisibility is not altered by a cyclical permutation of the digits of the original number.

3146. Proposed by L. H. BURNS, Student, Yale University.

Show, by elementary geometrical methods (preferably suitable for use in teaching a class in elementary solid geometry), that the volume of a regular icosahedron of edge a is

$$5a^3(3 + \sqrt{5})/12,$$

and obtain a similar formula for the volume of a regular dodecahedron of edge a.

3147. Proposed by N. MILLER, Queen's University, Canada.

If space of three dimensions be divided into cubes by means of three sets of equidistant parallel planes and a closed circuit be formed of their diagonals, what is the least number of diagonals necessary in order that this broken-line curve be knotted?

2672 [1918, 74]. Proposed by E. T. BELL, Seattle, Washington.

There is an identity in z, (1) $A(z) \equiv B(z)C(z)$; *e.g.*, $A(z) = 1/(1 - k^2z^2)$; $B(z) = 1/(1 - kz)$; $C(z) = 1/(1 + kz)$; and the formal expansions $A(z) = \Sigma a(n)z^n$, $B(z) = \Sigma b(n)z^n$, $C(z) = \Sigma c(n)z^n$,

$(n = 0, 1, \cdots, \infty)$, when substituted in (1), give, on equating coefficients:

$$a(n) = b(n)c(0) + b(n-1)c(1) + \cdots + b(0)c(n). \tag{2}$$

If (2) is an identity in n, justify such a use of non-convergent series to obtain it (*e.g.*, for $|k| \geqq 1$ in the above). This method of finding important identities (2) has been freely used by Hermite and many others without question of its validity, and without offering independent proofs of (2).

2694 [1918, 170]. Proposed by N. P. PANDYA, Sojitra, India.

Find the locus of the centroid of a triangle, whose vertex lies on a given parabola, whose base of given length is a segment of a given straight line of unlimited length, and one of whose base angles is known.

2723 [1918, 303]. Proposed by G. Y. SOSNOW, Newark, N. J.

The feet of the perpendiculars from the intersection of the diagonals on the sides of a cyclic quadrilateral M are joined to form a second quadrilateral N. Prove that N is a quadrilateral of minimum perimeter inscribed in M.

2724 [1918, 303]. Proposed by FRANK IRWIN, University of California.

Show that there is a unique set of real values, $x_1, x_2, x_3, \cdots, x_n$, that satisfy the equation

$$x_1^2 + x_2^2 + \cdots + x_n^2 - x_1x_2 - x_2x_3 - x_3x_4 - \cdots - x_{n-1}x_n - x_n + \frac{n}{2(n+1)} = 0.$$

SOLUTIONS.

3051 [1924, 49]. Proposed by NORMAN ANNING, University of Michigan.

Given the sequence: $u_1 = 2$, $u_2 = 8$, $u_n = 4u_{n-1} - u_{n-2}$, $(n = 3, 4, 5, \cdots)$, show that

$$(\pi/12) = \sum_{n=1}^{n=\infty} \operatorname{arccot} u_n^2.$$

SOLUTION BY THE PROPOSER.

It will first be shown that

$$u_n^2 - u_{n+1}u_{n-1} = 4. \tag{1}$$

For, when n is any integer greater than 2, we have

$$\begin{aligned} u_n(4u_{n-1}) &= u_{n-1}(4u_n) \\ u_n(u_n + u_{n-2}) &= u_{n-1}(u_{n+1} + u_{n-1}) \\ u_n^2 - u_{n+1}u_{n-1} &= u_{n-1}^2 - u_nu_{n-2} \\ &= u_{n-2}^2 - u_{n-1}u_{n-3} = \cdots \\ &= u_2^2 - u_3u_1 = 64 - 60 = 4. \end{aligned}$$

Now

$$\begin{aligned} \operatorname{arccot} u_n^2 &= \operatorname{arccot} [u_n(4u_n)/4] \\ &= \operatorname{arccot} [u_n(u_{n+1} + u_{n-1})/(u_n^2 - u_{n+1}u_{n-1})], \qquad \text{by (1),} \\ &= \operatorname{arccot} (u_{n+1}/u_n) - \operatorname{arccot} (u_n/u_{n-1}). \end{aligned} \tag{2}$$

If, in (2), we allow n to range from 2 to r and take the sum, we have

$$\operatorname{arccot} u_1^2 + \sum_{n=2}^{n=r} \operatorname{arccot} u_n^2 = \operatorname{arccot} (u_{r+1}/u_r). \tag{3}$$

As r approaches ∞, the ratio (u_{r+1}/u_r) approaches the larger root of the quadratic equation: $x^2 = 4x - 1$, namely, $2 + \sqrt{3}$. So

$$\sum_{n=1}^{n=\infty} \operatorname{arccot} u_n^2 = \operatorname{arccot} (2 + \sqrt{3}) = \pi/12.$$

3090 [1924, 353]. Proposed by H. S. UHLER, Yale University.

Show that the volume of the (smallest) segment of a sphere (radius = c) cut out by two mutually perpendicular planes, the distances of which from the center are a and b respectively, may be expressed by the formula

$$\frac{2}{3}c^3\cos^{-1}\left[\left(\frac{a}{\sqrt{c^2-b^2}}\right)\left(\frac{b}{\sqrt{c^2-a^2}}\right)\right]-\frac{1}{3}b(3c^2-b^2)\cos^{-1}\left(\frac{a}{\sqrt{c^2-b^2}}\right)$$
$$-\frac{1}{3}a(3c^2-a^2)\cos^{-1}\left(\frac{b}{\sqrt{c^2-a^2}}\right)+\frac{2}{3}ab\sqrt{c^2-(a^2+b^2)}.$$

[*Note.* This formula may be used as major part of an alternative solution of problem **2947** [*1922*, 29] as given by J. B. Reynolds [*1923*, 209].]

SOLUTION BY THE PROPOSER.

If the origin is taken at the center of the sphere, the required volume is

$$v=2\int_a^{\sqrt{c^2-b^2}}dy\int_b^{\sqrt{c^2-y^2}}dz\sqrt{c^2-y^2-z^2},\qquad(a^2+b^2\leqq c^2).$$

Performing the indicated integrations, we obtain the result given in the statement of the problem.

To apply this result to problem **2947**, we replace a, b, and c by $(a-h)$, $\sqrt{r^2-a^2}$, and r respectively, double the resulting volume, and add $l\left\{a^2\cos^{-1}\left(\frac{a-h}{a}\right)-(a-h)\sqrt{h(2a-h)}\right\}$, the volume of the cylindrical segment.

3095 [1924, 401]. Proposed by GEORGE RUTLEDGE, Massachusetts Institute of Technology.

Establish the following n identities involving the binomial coefficients of any even order:

$$1!\left\{\frac{1}{1}\binom{1}{1}\binom{2n}{n-1}-\frac{1}{2}\binom{2}{1}\binom{2n}{n-2}+\frac{1}{3}\binom{3}{1}\binom{2n}{n-3}-\cdots+(-1)^{n-1}\frac{1}{n}\binom{n}{1}\binom{2n}{0}\right\}=\frac{1}{2}\binom{2n}{n}.$$

$$3!\left\{\frac{1}{2}\binom{3}{3}\binom{2n}{n-2}\sum^{2}_{\binom{n-1}{1}}-\frac{1}{3}\binom{4}{3}\binom{2n}{n-3}\sum^{3}_{\binom{n-1}{1}}+\cdots+(-1)^{n-2}\frac{1}{n}\binom{n+1}{3}\right.$$
$$\left.\times\binom{2n}{0}\sum^{n}_{\binom{n-1}{1}}\right\}=\frac{1}{2}\binom{2n}{n}.$$

$$5!\left\{\frac{1}{3}\binom{5}{5}\binom{2n}{n-3}\sum^{3}_{\binom{n-1}{2}}+\cdots(-1)^{n-3}\frac{1}{n}\binom{n+2}{5}\binom{2n}{0}\sum^{(n)}_{\binom{n-1}{2}}\right\}=\frac{1}{2}\binom{2n}{n}.$$

. . . . . .

$$(2n-1)!\left\{\frac{1}{n}\binom{2n-1}{2n-1}\binom{2n}{0}\sum^{(n)}_{\binom{n-1}{i-1}}\right\}=\frac{1}{2}\binom{2n}{n}.$$

The notation $\sum^{(j)}_{\binom{n-1}{i-1}}$ is used to indicate the sum of the $\binom{n-1}{i-1}$ products of the squared reciprocals of the first n integers excepting j, taken $i-1$ at a time.

The first of these identities is well known in the form

$$\binom{2n}{0}-\binom{2n}{1}+\binom{2n}{2}-\cdots+\binom{2n}{2n}=0$$

and the last one is obvious.

SOLUTION BY THE PROPOSER.

It will suffice to illustrate the general case if the third identity is established. Consider the determinant

$$D = \begin{vmatrix} 1 & 2^2 & \cdots & n^2 \\ 1 & 2^4 & \cdots & n^4 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 1 & 2^{2n} & \cdots & n^{2n} \end{vmatrix}. \tag{1}$$

Multiply the first and second rows, respectively, by the coefficients of x^2 and x^4 in the polynomial $x^2(x^2-1)(x^2-2^2)$, and add the products to the third row. The first two elements of this row then reduce to zero and the kth ($k > 2$) element becomes

$$\frac{5!}{k}\binom{k+2}{5}k^2. \tag{2}$$

By an extension of this process all the elements below the principal diagonal may be reduced to zero. The value of the determinant is then found to be $n!3!5!\cdots(2n-1)!$. If A_{ij} is the cofactor of the element j^{2i} in the transformed determinant, then

$$\frac{1}{2}\binom{2n}{n}\frac{A_{2j}}{D} = (-1)^{3+j}\binom{2n}{n-j}\frac{1}{j^2}\sum^{(j)}\binom{n-\frac{1}{2}}{i}.$$

(See Pascal, *Die Determinanten*, p. 131. Also, *Journal of Mathematics and Physics of the Mass. Inst. of Tech.*, vol. 2, 1922, p. 47.) The development of the determinant in terms of the elements of the third row, using the reduced form in which the first two elements are zero and the values of the cofactors above, gives the desired identity. The proposer hopes that some reader of the MONTHLY may observe a simpler method for obtaining these identities, or an evaluation of the expressions $\sum^{(j)}\binom{n-1}{i-1}$.

3108 [1925, 46]. [Corrected.] Proposed by M. KURTZ, New York City.

Prove, or disprove, that

$$a^n = n! + n(a-1)^n - \frac{n(n-1)}{1\cdot 2}(a-2)^n + \cdots + (-1)^{n+1}(a-n)^n$$

holds for all values of n. Also prove or disprove the following corollary:

$$n(1^x) - \frac{n(n-1)}{1\cdot 2}(2^x) + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}(3^x) + \cdots + (-1)^{n-1}(n^x) = y,$$

where $y = 1$ when $x = 0$, $y = 0$ when $x = 1, 2, 3, \cdots(n-1)$, and $y = (-1)^{n+1}n!$ when $x = n$, both x and n being integral.

I. SOLUTION BY LOUIS WEISNER, University of Rochester.

The first equation may be written in the form

$$(1 - e^{-D})^n a^n = n! \tag{1}$$

where $D = d/da$ and $e^{hD}f(a) = f(a+h)$. From the generalisation of Leibniz's theorem:

$$F(D)uv = uF(D)v + Du\cdot F'(D)v + \frac{D^2u}{2!}\cdot F''(D)v + \cdots,$$

we have

$$\begin{aligned}(1-e^{-D})^{n+1}a\cdot a^n &= a(1-e^{-D})^{n+1}a^n + (n+1)e^{-D}(1-e^{-D})^n a^n \\ &= [a(1-e^{-D}) + (n+1)e^{-D}](1-e^{-D})^n a^n \\ &= [a(1-e^{-D}) + (n+1)e^{-D}]n! \\ &= (n+1)!,\end{aligned}$$

since $e^{hD} \cdot c = c$, where c is a constant. Hence if (1) is true for a given value of n, it is true when n is replaced by $n + 1$. But (1) is evidently true when $n = 1$. It follows by the usual argument employed in mathematical induction that (1) is true for all positive integral values of n.

From (1) it follows that

$$(1 - e^{-D})^{n+m} a^n = 0, \qquad m, n > 0.$$

Changing n to x and $n + m$ to n, we have

$$(1 - e^{-D})^n a^x = 0, \qquad n > x > 0.$$

Expanding and putting $a = 0$, we have the corollary as stated for $x = 1, 2, \cdots, (n - 1)$. From the expansion of $(1 - 1)^n$ by the binomial theorem it follows that when $x = 0$, $y = 1$; and from (1) it follows that when $x = n$, $y = (-1)^{n+1} n!$. Hence the theorem and corollary are correct as stated.

II. Solution by Harry Langman, New York City.

If u_x is a function of x, then

$$\Delta^n u_x = \sum_{r=0}^{n} (-1)^r C_r^n u_{x+n-r}.$$

If $u_x = x^n$, $\Delta^n u_x = n!$. Hence

$$\Delta^n u_{a-n} = n! = \sum_{r=0}^{n} (-1)^r C_r^n (a - r)^n,$$

which is the required identity. Since this is true for all values of a, successive differentiation with respect to a gives

$$0 = \sum_{r=0}^{n} (-1)^r C_r^n (a - r)^x, \qquad 0 \leqq x \leqq n - 1.$$

By setting $a = 0$ we obtain the results in the corollary.

Also solved by H. Betz, J. Ginsburg, C. A. Shook, and H. L. Smith.

3109 [1925, 46]. Proposed by the late J. W. Nicholson.

Find two rational numbers which separate the roots of the equation $x^3 - ax^2 + bx - c = 0$.

Solution by C. F. Gummer, Queen's University.

It will be understood that two rational functions of the coefficients $f(a, b, c)$ and $g(a, b, c)$ are to be found, such that, whenever the equation has three real unequal roots $x_1 < x_2 < x_3$, either $x_1 < f \leqq x_2 \leqq g < x_3$ or $x_1 < g \leqq x_2 \leqq f < x_3$.

If instead we attempt to make $x_1 < f < x_2 < g < x_3$ in every case, we find that the problem admits no solution. For, in the first place, it is readily seen, by means of a similarity transformation, that such an f, when expressed in terms of the roots, must be a rational homogeneous expression of degree unity (whether integral or fractional); and hence that f is replaced by $-f$ when x_1, x_2, and x_3 are replaced by $-x_1$, $-x_2$, and $-x_3$. If, therefore, x_1 changes continuously to $-x_3$, x_2 to $-x_2$, and x_3 to $-x_1$, which does not necessitate any two roots becoming equal at any stage, f will be found to have changed from a value in the interval between the two smaller roots to a value between the two greater. It is therefore impossible to find a suitable function f which will remain always in the left-hand interval.

It is, however, easy to select an f which remains always in the greater of the two intervals (x_1, x_2) and (x_2, x_3); except when these intervals are equal. For we may take

$$f = a/3 = (x_1 + x_2 + x_3)/3 = x_2 + (x_1 + x_3 - 2x_2)/3.$$

It remains to choose a g which shall be restricted to the smaller interval between the roots. This must be a symmetric function of the roots expressible in the form $x_2 - (x_1 + x_3 - 2x_2)U/V$, where V is a symmetric polynomial in the x's and U is a polynomial such that U/V is positive when x_2 is the middle root. If either x_1 or x_3 approaches x_2, g must approach x_2 also. Hence U is divisible by $x_2 - x_1$ and by $x_3 - x_2$.

In seeking the simplest possible form for g, we may try

$$U = (x_2 - x_1)(x_3 - x_2), \qquad V = p(x_1^2 + x_2^2 + x_3^2) + q(x_1x_2 + x_1x_3 + x_2x_3).$$

The conditions for the symmetry of g show that p must be 2 and q must be -2. This makes V a positive definite quadratic form in the roots, so that U/V has the required sign. Hence a suitable expression for g is

$$(\Sigma x_1^2x_2 - 6x_1x_2x_3)/(2\Sigma x_1^2 - 2\Sigma x_1x_2) = (ab - 9c)/(2a^2 - 6b).$$

The conclusion is that *if the roots of the cubic* $x^3 - ax^2 + bx - c = 0$ *are real and distinct and not in arithmetic progression, they are separated by* $a/3$ *and* $(ab - 9c)/(2a^2 - 6b)$; *and* $a/3$ *lies in the greater interval.*

There are no three rational functions of the coefficients of the general quartic which separate the roots in this manner. For if there were, one of the functions would approach x_1 as x_2 approached x_1. It would therefore take the form $x_1 + (x_2 - x_1)P/Q$, Q being a symmetric function not divisible by the differences of roots. By symmetry, the same quantity would be expressible in the form $x_3 + (x_4 - x_3)R/Q$, which leads to a contradiction when x_2 approaches x_1 and x_4 approaches x_3 at the same time.

It is another question, however, whether or not there exists a rational algebraic process to determine numbers separating the roots of the general numerical quartic.

REMARK BY OTTO DUNKEL, Washington University.

To show that g lies in the smaller interval, it suffices to show that

$$2[\Sigma x_1^2 - \Sigma x_1x_2]x_1 < \Sigma x_1^2x_2 - 6x_1x_2x_3 < 2[\Sigma x_1^2 - \Sigma x_1x_2]x_3.$$

The first inequality reduces to

$$(x_3 - x_1 + x_2 - x_1)(x_3 - x_1)(x_2 - x_1) > 0,$$

and the second to

$$(x_3 - x_1 + x_3 - x_2)(x_3 - x_1)(x_3 - x_2) > 0.$$

Both of these inequalities are true if $x_1 < x_2 < x_3$.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to R. W. BURGESS, c/o Western Electric Co., 195 Broadway, New York City.

The Franklin Institute has conferred Franklin medals and certificates of honorary membership on Dr. P. ZEEMAN, professor of physics at the University of Amsterdam, and Dr. ELIHU THOMSON, of the General Electric Company.

J. R. FREEMAN, consulting engineer, of Providence, has made a gift of securities valued at $25,000 to the Boston Society of Civil Engineers for the establishment of a fund, the income of which is to be used for encouraging research by the younger engineers of the Society through the award of prizes for papers on hydraulics and allied subjects.

Professor NIELS BOHR and Professor A. S. EDDINGTON have been elected foreign members of the National Academy of Sciences. Professor S. LEFSCHETZ, of Princeton University, has been elected a member in the section of mathematics.

Professor G. G. CHAMBERS has received the honorary degree of doctor of science from Dickinson College.

Columbia University has conferred the honorary degree of doctor of science on Dr. IRVING LANGMUIR, of the General Electric Company.

Yale University has conferred the honorary degree of doctor of science on Professor R. A. MILLIKAN.

The Case School of Applied Science, and Union College have conferred honorary degrees on Professor M. I. PUPIN, of Columbia University.

Assistant Professor J. R. KLINE, of the University of Pennsylvania, has been awarded a Guggenheim Fellowship to study the analysis situs of three dimensions from a point-set standpoint, principally at the University of Göttingen.

Assistant Professor J. L. WALSH, of Harvard University, has been awarded a National Research Fellowship.

At West Virginia University, Assistant Professors MARGARET BUCHANAN, C. N. REYNOLDS, and B. M. TURNER have been promoted to associate professorships of mathematics.

Assistant Professor J. E. DAVIS, of the Drexel Institute, has been appointed associate professor of engineering extension at Pennsylvania State College.

Dr. PHILIP FRANKLIN has been promoted to an assistant professorship of mathematics at the Massachusetts Institute of Technology.

Dr. C. C. GROVE has been appointed assistant professor of mathematics at the Brooklyn Polytechnic Institute.

Associate Professor HILLEL HALPERIN has been promoted to a full professorship of mathematics at the Agricultural and Mechanical College of Texas.

Miss FRANCES HARSHBARGER, of West Virginia University, has been appointed head of the department of mathematics at Potomac State College, Keyser, W. Va.

Professor A. A. MICHELSON, of the University of Chicago, has been appointed to the first of the distinguished service fellowships recently established at that university.

Assistant Professor E. J. MILES, of Yale University, has been promoted to an associate professorship of mathematics.

Mr. MAX MORRIS has been promoted to an assistant professorship of mathematics at the Case School of Applied Science.

Miss ELIZABETH STAFFORD has been appointed adjunct professor of mathematics at the West Texas College of Technology, Lubbock.

Dr. J. H. TAYLOR has been appointed assistant professor of mathematics at Lehigh University.

Associate Professor H. S. UHLER, of Yale University, has been appointed head of the department of physics at Gettysburg College.

Professor H. A. WILSON, of the University of Glasgow, has accepted reappointment to the professorship of physics at Rice Institute which he held from 1912 to 1924.

Professor FREDRICK WOOD, of Lake Forest College, has been appointed professor of mathematics at Wesleyan College, Macon, Ga.

Mr. E. P. STARKE has been promoted to an assistant professorship of mathematics at Rutgers College.

Professor G. N. ARMSTRONG, of Ohio Wesleyan University, has been appointed head of the department of mathematics to fill the vacancy caused by the death of Professor C. B. AUSTIN.

At the University of Colorado, Professor I. M. DELONG has been made professor emeritus after forty-seven years of service. Professor A. J. KEMPNER, of the University of Illinois, has been appointed professor of mathematics and acting head of the department.

At Wesleyan College, Macon, Ga., Professor J. C. HINTON has ceased active work after thirty-five years of service. He has been dean of the college for the past twenty-five years.

Professor C. N. MILLS, of Aberdeen, South Dakota, has been appointed head of the mathematics department of the Illinois State Normal University.

Professor L. E. DICKSON, of the University of Chicago, would appreciate information as to where he could borrow or buy G. B. Jerrard's *Mathematical Researches, Part II*, 1834.

The following appointments to instructorships are announced: American College for Women, Constantinople, E. MARIE PLAPP; Connecticut College for Women, MILDRED E. CARLEN; University of Florida, CLAIRE HARKINS; West Virginia University, H. A. DAVIS; University of Michigan, B. DUSHNIK, N. C. FISKE, J. D. GRANT.

Dr. W. S. DENNETT, of New York City, died March 6, 1925. He was treasurer of the American Mathematical Society from 1900 to 1907.

Professor W. A. HAMILTON, of Antioch College, died June 25, 1925.

Professor S. J. LOCKNER, of the University of Pittsburgh, died May 10, 1925, at the age of fifty-five.

Dr. MANSFIELD MERRIMAN, professor of civil engineering at Lehigh University from 1878 to 1907, died June 7, 1925, at the age of seventy-seven.

Professor FELIX KLEIN, of the University of Göttingen, died June 22, 1925, at the age of seventy-six.

THE SURNAMED CHOSEN CHEST.

By DAVID EUGENE SMITH, Columbia University.

II. Orientalia.

In the "Surnamed Chosen Chest" [1] there are many oriental manuscripts and early printed books,—some hundreds in all. It is proposed in this article to mention a few of the most interesting manuscripts of the mathematical classics of the East. There are not many copies of these works in this country, and it is probable that a brief list of some of the more important ones will be of service to scholars, not merely in the history of mathematics but in the field of oriental languages as well. The material is available for study, through photostat copies, by anyone who cares to have such copies made and who is prepared to undertake work of this nature.

Referring first to the Hindu classics, the anonymous *Sūrya Siddhānta* was the earliest noteworthy Indian work on astronomy. It is mathematically interesting because of its early trigonometric table,[2] and there is in the library a remarkably well-preserved copy on palm leaves. The text is Sanskrit, 78 leaves, $1\frac{1}{2}$ by 13 in.[3] The original work was written *c.* 400, and this copy was made *c.* 1450. Although slightly damaged, the text seems to be substantially complete.

The first of the prominent Hindu mathematicians whose names are known was Āryabhāta (*c.* 500), and there is in the library a manuscript of the entire *Āryabhatīya* (*Āryabhatīyam*). This is on palm leaves, 2 by 19 in., 76 leaves (152 pages). It is a modern copy of the Cadjan manuscript in Madras.

Of the native astronomers of India, those who joined mathematics with the science of the stars, the best-known is Varāhamihira (*c.* 505). There is in the library a complete manuscript on paper of his great *Pañca Siddhāntikā*, a work of much rarity in this complete form and consisting of several hundred leaves. The text is Sanskrit, and the earliest portions of the copy date from the sixteenth century.

The leading mathematician of India during the Middle Ages was Bhāskara (1114–*c.* 1185), a native of Biddur in the Deccan, but working at the great astronomical center Ujjain.[4] Of his various works the one which brought him the most fame is the *Lilāvati*, a treatise on arithmetic and mensuration. It was translated into Persian in 1587 by Fyzi, and the latter's story of Bhāskara's daughter is well known although not authenticated in Hindu sources. There are two translations in English.

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[1] The present article forms a continuation of that which appeared in this volume, pages 287–294.

[2] See the author's *History of Mathematics*, I, 34, 145; II, 625; hereafter referred to as *History*.

[3] These measurements are merely approximate, as is necessarily the case with all old manuscripts.

[4] *History*, I, 275.